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Algebraic Rieffel induction, formal Morita equivalence, and applications to deformation quantization

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Abstract

In this paper, we consider algebras with involution over a ring C which is given by the quadratic extension by i of an ordered ring R. We discuss the *-representation theory of such *-algebras on pre-Hilbert spaces over C and develop the notions of Rieffel induction and formal Morita equivalence for this category analogously to the situation for C^*-algebras. Throughout this paper, the notion of positive functionals and positive algebra elements will be crucial for all constructions. As in the case of C^*-algebras, we show that the GNS construction of *-representations can be understood as Rieffel induction and, moreover, that formal Morita equivalence of two *-algebras, which is defined by the existence of a bimodule with certain additional structures, implies the equivalence of the categories of strongly non-degenerate *-representations of the two *-algebras. We discuss various examples like finite rank operators on pre-Hilbert spaces and matrix algebras over *-algebras. Formal Morita equivalence is shown to imply Morita equivalence in the ring-theoretic framework. Finally, we apply our considerations to deformation theory and in particular to deformation quantization and discuss the classical limit and the deformation of equivalence bimodules. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and motivation

In this work, we discuss algebras with a *-involution over ordered rings, study their representation theory, and develop tools analogously to the well-known case of C^*-algebras. Our main motivation comes from deformation quantization, where the star product algebras still have a *-involution but no topological structure like a C^*-norm, and there are further examples and applications both in physics and mathematics. We start with an ordered ring R and its quadratic ring extension $\mathsf{C} = \mathsf{R}(\mathrm{i})$, where $\mathrm{i}^2 = -1$, and consider *-algebras over C. The interplay between the ordering structure in R and the *-involution gives rise to various notions of positivity which make up the heart of this paper. We consider, for a *-algebra over C, the category of *-representations on pre-Hilbert spaces over C and find that positivity and *-involution together are sufficiently powerful tools which enable us to formulate many results known from C^*-algebras in this purely algebraic framework. Following the general idea of concentrating on the algebraic properties of C^*-algebras, we consider in this paper analogues of Rieffel induction and Morita equivalence as well as various aspects of formal deformation theory related to these constructions.

The concept of Morita equivalence has been applied to many different categories in mathematics, and its main goal is to explore the relationship between 'objects' and their 'representation theory', i.e. their 'theory of modules'. This idea was first made precise in a purely algebraic context, the category of unital rings, by Morita, see [7,55,56]: two unital rings are called Morita equivalent if their categories of left modules are equivalent [54]. The main result of this theory states that Morita equivalent rings always come with a pair of corresponding bimodules of a certain type in such a way that the functors implementing the equivalence of the categories are actually equivalent to tensoring with these bimodules. Morita equivalent rings share many ring theoretical properties, the 'Morita invariants', like Hochschild cohomology and algebraic K-theory and properties like being Artinian, semi-simple, or Noetherian, see [2,42,50]. They also have isomorphic lattices of ideals and isomorphic centers. It follows that commutative unital rings are Morita equivalent if and only if they are isomorphic, hence Morita equivalence is most interesting if at least one of the rings is non-commutative. Commutativity is not Morita invariant and in fact, the classical example of Morita equivalent rings is given by a unital ring R and the corresponding matrix ring $M_n(\mathsf{R})$.

Since then, the notion of Morita equivalence has been adapted to many other algebraic contexts, such as non-unital rings [1,3,61], monoids [6,48], coalgebras [52] as well as to more topological and geometric settings, as for example topological groupoids [57], symplectic groupoids and Poisson manifolds [79,80]. In a recent work, Ara [5] defined the notion of Morita equivalence for rings with involution, which is related to the approach developed in the present paper (see note added to the end of Section 10).

In the context of C^*-algebras, the 'theory of modules' is given by *-representations on Hilbert spaces. Here Rieffel defined the notion of (strong) Morita equivalence and induced representations motivated mainly by the theory of induced representations of locally compact groups by closed subgroups [65,66]. In particular, a new and simpler proof of Mackey's imprimitivity theorem [53] was given in terms of group C^*-algebras, see [65] and also [51,64] for further discussions and applications. A related approach was developed by Fell in [41] for Banach algebras, which also led to a proof of Mackey's imprimitivity theorem through group algebras. The fundamental notion of induced representations of C^*-algebras, now called 'Rieffel induction', is the C^*-algebra analogue of the older idea of constructing functors between categories of modules over rings R and S by means of tensoring with an (R–S)-bimodule. In this purely algebraic setting, functors arising in this way are rather general and, in fact, any equivalence of categories must be of this type (see [7,76]). For Rieffel's induction, one has to add additional structures to the bimodule in order to end up again with a representation on a Hilbert space. This induction of *-representations as well as the notion of Morita equivalence have become important tools in the study of C^*-algebras, and Morita equivalence is now one of the most important equivalence relations in this category, see also [64,67] and references therein. Moreover, both Rieffel induction and Morita equivalence of C^*-algebras have been used in various fields of physics like quantization and phase space reduction [51]; they also arise in the context of applications of non-commutative geometry to string and M theory [30,69,71,72].

On the other hand, there are many situations in mathematics and physics where interesting algebras occur which are not C^*-algebras and where no obvious embedding into a C^*-algebra is available. The canonical commutation relations $[q, p] = \mathrm{i}\hbar$ are known to be incompatible with a representation by bounded operators and, more generally, the commutation relations in the universal enveloping algebra of a Lie algebra typically exhibit this behavior. While in this case one can obtain bounded operator representations by passing to unitary group representations, in the more general case of q-deformed universal enveloping algebras it is less evident whether one can 'exponentiate' in a meaningful way to obtain bounded operator representations, see, e.g., [47] and references therein. Another typical example is given by the algebra of (pseudo-)differential operators on a manifold. Certain subspaces of pseudo-differential operators define *-algebras, where the *-involution can be induced by their action on the smooth functions with compact support equipped with a Hermitian product given by a positive density, see, e.g., [15]. These operators are continuous with respect to certain locally convex topologies of smooth functions, but they are typically unbounded with respect to the operator norm induced by the pre-Hilbert space structure of smooth functions with compact support. Finally, closely related to this situation, our main example is given by deformation quantization as introduced by Bayen et al. [9]; see also [73,77] for recent surveys. In this quantization scheme, the classical observable algebra is given by the complex-valued smooth functions on a symplectic, or, more generally, on a Poisson manifold and the pointwise product is deformed into a $\hbar$-dependent associative product, the star product, such that in zeroth order of Planck's constant $\hbar$, the star product equals the pointwise product and in the first order the commutator yields i times the Poisson bracket. The star product is usually considered as a formal power series in $\hbar$ so one ends up with a formal deformation in the sense of Gerstenhaber [42]. Thus, here the underlying ground ring is changed from $\mathbb{R}$ and $\mathbb{C}$ to $\mathbb{R}[[\hbar]]$ and $\mathbb{C}[[\hbar]]$, respectively. In addition, we shall always assume that the function 1 is still the unit element with respect to the star product and that the star product is bidifferential, a feature which is usually fulfilled and has various important consequences concerning in particular the representation theory [75]. In the symplectic case, the existence of star products was first shown by DeWilde and Lecomte [32], then independently by Fedosov [38,39], who gave a recursive construction, and Omori et al. [60]. In the more general case of Poisson manifolds Kontsevich has shown this existence [49]. The classification up to cohomological equivalence is due to Nest and Tsygan [58,59], Bertelson et al. [11], Deligne [31], Weinstein and Xu [78], and Kontsevich [49].

Common to all the above examples of associative algebras is that they all have a *-involution: this is obvious in the case of a complexified universal enveloping algebra of a real Lie algebra and for (pseudo-)differential operators, and it can also be achieved by some additional requirements for star products. Since no C^*-norm or similar topological structures are present, we shall investigate *-algebras from the algebraic point of view only (see [4,5] for a related approach). On the other hand, there is a notion of *positivity* in the underlying ground ring which is evident for $\mathbb{R}$, but also the formal power series $\mathbb{R}[[\lambda]]$ with real coefficients is an ordered ring. This positivity can be understood in an 'asymptotic' sense which fits very well into the formal character of the star products. The star products can be understood heuristically as 'asymptotic expansions' of a strict deformation quantization as formulated by Rieffel [68], see also Landsman's book [51], even if it is not clear whether such a strict counterpart exists or not. On the other hand, it is clear from the physical point of view that the formal character is not sufficient for a reasonable quantization. Thus one has to deal with the problem of convergence of the formal star products. Starting in the formal framework, this difficult question is usually attacked by considering suitable subalgebras, see, e.g., [13,15–17,24–27] and references therein. These investigations provide at least in some cases a bridge between formal and strict deformation quantization. This motivates the idea that the asymptotic point of view in formal deformation quantization already contains most of the important information needed for quantization. We are then led to the program of finding 'formal' or 'asymptotic' analogues of various techniques and results known from C^*-algebra theory and applying them in a more algebraic framework, as in deformation quantization. Certainly, this is of great interest if one wants to understand the classical and semi-classical limits of these constructions but is not necessarily restricted to quantization, as the formal parameter can correspond to other quantities like a coupling constant [34,35]. One can also think of investigations of Connes' non-commutative geometry [29] from the asymptotic point of view. This all motivates us to consider *-algebras over ordered rings in general.

In fact, several interesting results following this program have already been obtained, starting with the investigation of the GNS construction in the formal case by Bordemann and Waldmann [20,21]. Here the ordering structure of an ordered ring allows one to define positive linear functionals of *-algebras as in the C^*-algebra case which leads to the analogue of the well-known GNS construction of *-representations, see, e.g., [22,29,43]. It was shown in [20,21] that this concept leads to a physically reasonable representation theory for star products and has been extended and applied to various situations like deformation quantization on cotangent bundles with the presence of a cohomologically non-trivial magnetic field, i.e. a monopole [15], the WKB approximation [16,17,20], and thermodynamical KMS states and their representations [18,19] including a formal Tomita–Takesaki theory [75].

In this paper we set up the general framework of *-representations of *-algebras over ordered rings, develop the notions of algebraic Rieffel induction and formal Morita equivalence, and apply our results to deformations of *-algebras. In detail, we have obtained the following results:

In Section 2, we discuss elementary properties of ordered rings, pre-Hilbert spaces, *-algebras and their *-representations, as well as the definition of positive algebra elements and approximate identities. The concept of a *-algebra with sufficiently many positive linear functionals turns out to be important. In this case, one obtains faithful pre-Hilbert space representations and also nicer algebraic properties, like no non-zero normal nilpotent elements. Moreover, such algebras are torsion-free, see Proposition 2.8.

In Section 3, we consider bimodules with inner products which take their values in a *-algebra and use such bimodules to obtain a purely algebraic version of Rieffel induction in Theorem 3.5. Here everything is analogous to the case of C^*-algebras except for the important additional positivity requirement (P) which will be crucial throughout this paper. We discuss some different and easier-to-use conditions (P1)–(P3) and (PC) which imply (P), see Lemma 3.1.

Section 4 is devoted to various standard constructions related to Rieffel induction which we shall need in the sequel. We consider direct sums in Lemma 4.1, tensor products in Proposition 4.5 and the commutant of *-representations in Proposition 4.2. We also discuss how to use homomorphisms to construct bimodules with the needed inner products, see Proposition 4.8. Furthermore, we show that the GNS construction of a representation can be viewed as a particular case of Rieffel induction, see Proposition 4.7.

In Section 5, we develop the notion of an equivalence bimodule for two *-algebras, which is a bimodule together with two inner products, one for each *-algebra, with some compatibility properties (see Definition 5.1). Two *-algebras are called formally Morita equivalent if there exists such an equivalence bimodule, see Definition 5.3. We discuss reflexivity and transitivity properties (Propositions 5.4 and 5.6) of this relation and define the notion of a non-degenerate equivalence bimodule in Definition 5.9. The existence of a non-degenerate equivalence bimodule then implies the equivalence of the categories of strongly non-degenerate *-representations, see Theorem 5.10. An example using the Grassmann algebra shows that the converse is not true in general (Corollary 5.20), as the property of having sufficiently many positive linear functionals is preserved by formal Morita equivalence, see Proposition 5.19. Finally, we consider the question of how to construct a non-degenerate equivalence bimodule out of an equivalence bimodule in Proposition 5.22.

Section 6 contains the main examples. First we introduce the notion of finite rank operators on a right module analogous to the compact operators in the C^*-algebra case and show that for an equivalence bimodule the first algebra is isomorphic to the finite rank operators on the equivalence bimodule with respect to the right module structure of the other algebra, see Proposition 6.1. Next we consider the direct sum $\mathsf{C}^{(\Lambda)}$, where Λ is an arbitrary index set and use this as a C-right module and as a left module for the finite rank operator $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ on C to show the formal Morita equivalence of C and $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ and in particular the formal Morita equivalence of $M_n(\mathsf{C})$ and C in Proposition 6.10 and generalize this to arbitrary pre-Hilbert spaces in the case of an ordered field (Proposition 6.8). Considering tensor products of bimodules and the underlying *-algebras in Proposition 6.9, we arrive in particular at the formal Morita equivalence of a *-algebra $\mathcal{A}$ and $M_n(\mathcal{A})$ provided $\mathcal{A}$ has an approximate identity, see Proposition 6.7. Finally, we consider full projections in Propositions 6.12 and 6.14.

In Section 7, we specialize to unital *-algebras and prove that formal Morita equivalence implies Morita equivalence in the sense of unital rings, see Corollary 7.3, and we also show that the converse is not true in general. As a consequence, we prove that the centers of formally Morita equivalent *-algebras are *-isomorphic, see Proposition 7.6, and we apply this result to algebras of smooth functions, see Corollary 7.8. These results shall be used later in Section 9, in the context of deformation quantization.

In Section 8, we start to set up the framework of formal deformations of *-algebras and their *-representations. We consider formal associative deformations which allow in addition for a deformation of the *-involution. Then the important observation that $\mathsf{R}[[\lambda]]$ is still an ordered ring if R is ordered shows that we stay in the same framework of *-algebras over ordered rings. We discuss deformations of positive linear functionals, positive deformations of the *-algebras in the sense of [23, Definition 4.1], and deformations of approximate identities, and consider the corresponding classical limits. Moreover, we define the classical limit of a pre-Hilbert space over $\mathsf{C}[[\lambda]]$ and of *-representations, see Lemmas 8.2, 8.3 and Proposition 8.5.

We continue the discussion of deformations and classical limits in Section 9 by defining the classical limit of bimodules. We show that the classical limit is a bimodule for the classical limits of the corresponding algebras of the same type, see Proposition 9.4, and compute the relation of the corresponding functors of algebraic Rieffel induction in Proposition 9.5. Here the notion of positive deformations becomes crucial. In particular, formal Morita equivalence of the deformed algebras implies, under some technical assumptions, formal Morita equivalence of the classical limits, see Theorem 9.7. We conclude that for Morita equivalent star products, the underlying manifolds have to be diffeomorphic and give thereby an ‘asymptotic explanation’ why strongly Morita equivalent quantum tori must have at least the same classical dimension. Finally, we discuss the other direction, namely the deformation of (equivalence) bimodules with all their relevant structures and present one basic example using a deformation of a *-homomorphism in Proposition 9.11.

Section 10 contains a conclusion and several open questions related to our approach. In Appendix A we collect some elementary properties of positive matrices and in Appendix B we discuss positive functionals and elements for the algebra of smooth functions on a manifold.

Notation. The formal parameter will be denoted by λ and corresponds in deformation quantization in convergent situations directly to $\hbar$. As we shall need various tensor products we shall indicate the underlying ring sometimes as an index, but to avoid clumsy notation we shall omit this whenever possible.

2. Ordered rings, pre-Hilbert spaces and *-algebras

In this section we shall discuss some basic definitions and results on ordered rings as well as on pre-Hilbert spaces and *-algebras over such rings, see, e.g., [17,20,21,23], in order to find algebraic analogues of the corresponding constructions in C^*-algebra theory, see, e.g., the textbooks [22,29,43,51].

Let R be an associative, commutative ring with $1 \neq 0$ and let $\mathsf{P} \subset \mathsf{R}$. Then (R, P) is called an *ordered ring* with positive elements P if R is the disjoint union $\mathsf{R} = -\mathsf{P} \cup \{0\} \cup \mathsf{P}$ and for all $a, b \in \mathsf{P}$ one has $a+b, ab \in \mathsf{P}$. As usual we define $a > b$ if and only if $a-b \in \mathsf{P}$ and similarly '$<$', '$\geq$', and '$\leq$' which provides an ordering for the ring R. Then $a^2 > 0$ for all $a \neq 0$ and hence $1 > 0$. Moreover, R is of characteristic zero, since $n1 = 1 + \cdots + 1 > 0$, and R has no zero divisors. The corresponding quotient field $\hat{\mathsf{R}}$ of R inherits the ordering structure and becomes an ordered field by defining $\hat{\mathsf{P}} := \{a/b | ab \in \mathsf{P}\}$ and the inclusion $\mathsf{R} \hookrightarrow \hat{\mathsf{R}}$ preserves the order.

If R is an ordered ring, we consider $\mathsf{C} := \mathsf{R} \oplus \mathrm{i}\mathsf{R} = \mathsf{R}(\mathrm{i})$, where we endow C with the structure of an associative, commutative ring by requiring $\mathrm{i}^2 = -1$. This quadratic ring extension has again characteristic zero and no zero divisors. Elements in C are written as $z = a + \mathrm{i}b$ with $a, b \in \mathsf{R}$ and we can embed $\mathsf{R} \hookrightarrow \mathsf{C}$ by $a \mapsto a + \mathrm{i}0$. As in the case of complex numbers we define the complex conjugation in C by $z = a + \mathrm{i}b \mapsto \bar{z} := a - \mathrm{i}b$. Then $z \in \mathsf{C}$ is real if $z = \bar{z}$ which is the case if $z \in \mathsf{R} \subset \mathsf{C}$. Moreover, $\bar{z}z \geq 0$ and $\bar{z}z = 0$ if and only if $z = 0$.

Besides the real and complex numbers the basic example we have in mind is the formal power series with real and complex coefficients, where $\mathbb{R}[[\lambda]]$ is endowed with the canonical ring ordering by setting $a = \sum_{r=r_0}^{\infty} \lambda^r a_r > 0$ for $a \neq 0$ if $a_{r_0} > 0$, where $r_0 \in \mathbb{N}$ is the first index with non-vanishing coefficient. Note that this ordering is non-Archimedian since, e.g. $0 < n\lambda < 1$ for all $n \in \mathbb{N}$.

Consider an ordered ring R and the corresponding quadratic ring extension C and let $\mathfrak{H}$ be a C-module. A map $\langle \cdot, \cdot \rangle : \mathfrak{H} \times \mathfrak{H} \to \mathsf{C}$ satisfying

$$\langle \phi, a\psi + b\chi \rangle = a\langle \phi, \psi \rangle + b\langle \phi, \chi \rangle, \quad \overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle, \quad \text{and} \quad \langle \phi, \phi \rangle \geq 0 \tag{2.1}$$

for all $\phi, \psi, \chi \in \mathfrak{H}$ and $a, b \in \mathsf{C}$ is called a *semi-definite Hermitian product* for $\mathfrak{H}$. If $\langle \cdot, \cdot \rangle$ satisfies in addition to the non-degeneracy condition

$$\langle \phi, \phi \rangle = 0 \Rightarrow \phi = 0, \tag{2.2}$$

then $\langle \cdot, \cdot \rangle$ is called a *Hermitian product* and $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ is called a *pre-Hilbert space* over C. Note that we have used the physicists' convention of linearity in the second argument. From the non-degeneracy it follows that $\mathfrak{H}$ is torsion-free. A linear map $U : \mathfrak{H}_1 \to \mathfrak{H}_2$, where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are C-modules with semi-definite Hermitian products, is called *isometric* if $\langle U\phi, U\psi \rangle_2 = \langle \phi, \psi \rangle_1$ for all $\phi, \psi \in \mathfrak{H}_1$, and *unitary* if U is isometric and bijective. As usual we conclude that the inverse of a unitary map is unitary and an isometric map is automatically injective if $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are pre-Hilbert spaces.

Lemma 2.1. *Let $\mathfrak{H}$ be a C-module with semi-definite Hermitian product.*

1. *The Cauchy–Schwarz inequality*

$$\langle \phi, \psi \rangle \overline{\langle \phi, \psi \rangle} \leq \langle \phi, \phi \rangle \langle \psi, \psi \rangle \tag{2.3}$$

holds for all $\phi, \psi \in \mathfrak{H}$.

2. *The space* $\{\phi \in \mathfrak{H} | \langle \phi, \phi \rangle = 0\}$ *coincides with* $\mathfrak{H}^{\perp} := \{\phi \in \mathfrak{H} | \forall \psi \in \mathfrak{H} : \langle \phi, \psi \rangle = 0\}$ *which is a* C*-submodule of* $\mathfrak{H}$. *The quotient* $\mathfrak{H}/\mathfrak{H}^{\perp}$ *endowed with the Hermitian product* $\langle [\phi], [\psi] \rangle := \langle \phi, \psi \rangle$ *is a pre-Hilbert space over* C.

The proof is as in the case of complex numbers with the only technicality that we have to use the quotient fields $\hat{\mathsf{R}}$ and $\hat{\mathsf{C}}$ to prove (2.3). Nevertheless (2.3) holds in R, see also [21,75].

As we shall also need the degenerate case in the sequel, we shall now consider a C-module $\mathfrak{H}$ with semi-definite Hermitian product $\langle \cdot, \cdot \rangle$ more closely. For a given $A \in \mathrm{End}_{\mathsf{C}}(\mathfrak{H})$, we say that there exists an adjoint $B \in \mathrm{End}_{\mathsf{C}}(\mathfrak{H})$ if one has

$$\langle \phi, A\psi \rangle = \langle B\phi, \psi \rangle \tag{2.4}$$

for all $\phi, \psi \in \mathfrak{H}$. In this case B is called an *adjoint* of A. Analogously, one defines adjoints of maps $A \in \mathrm{End}_{\mathsf{C}}(\mathfrak{H}_1, \mathfrak{H}_2)$ for two C-modules $\mathfrak{H}_1, \mathfrak{H}_2$ with positive semi-definite Hermitian product. Next we define the spaces (cf., e.g., [75])

$$\mathfrak{B}(\mathfrak{H}) := \{A \in \mathrm{End}_{\mathsf{C}}(\mathfrak{H}) | A \text{ has an adjoint}\}, \tag{2.5}$$

$$\mathfrak{I}(\mathfrak{H}) := \{N \in \mathrm{End}_{\mathsf{C}}(\mathfrak{H}) | \mathrm{im} N \subseteq \mathfrak{H}^{\perp}\}, \tag{2.6}$$

and similarly $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. We obtain immediately the following lemma by a straightforward computation:

Lemma 2.2. *Let* $\mathfrak{H}$ *be a* C*-module with semi-definite Hermitian product and let* $A, B \in \mathfrak{B}(\mathfrak{H})$ *and* $a, b \in \mathsf{C}$. *Let* A^*, B^* *be adjoints of* A, B, *respectively*.

1. $aA + bB, AB \in \mathfrak{B}(\mathfrak{H})$ *and* $\bar{a}A^* + \bar{b}B^*, B^*A^*$ *are adjoints of* $aA + bB, AB$, *respectively*.
2. *For any adjoint* A^* *of* A *one has* $A^* \in \mathfrak{B}(\mathfrak{H})$ *and* A *is an adjoint of* A^*.
3. $\mathfrak{I}(\mathfrak{H}) \subset \mathfrak{B}(\mathfrak{H})$ *is a two-sided ideal of* $\mathfrak{B}(\mathfrak{H})$. *Any adjoint of* A *is of the form* $A^* + N$, *where* A^* *is a particular adjoint and* $N \in \mathfrak{I}(\mathfrak{H})$ *is arbitrary*.

Next we consider an associative algebra $\mathcal{A}$ over C. An involutive antilinear map ${}^* : \mathcal{A} \to \mathcal{A}$ is called a *-*involution* for $\mathcal{A}$ if for all $A, B \in \mathcal{A}$ one has $(AB)^* = B^*A^*$. An associative algebra over C equipped with such a *-involution is called *-*algebra* over C. As usual we define *Hermitian*, *normal*, *isometric* and *unitary elements* in $\mathcal{A}$.

Let $\mathcal{A}$ be a *-algebra and $\omega : \mathcal{A} \to \mathsf{C}$ a linear functional. Then ω is called *positive* if for all $A \in \mathcal{A}$ one has

$$\omega(A^*A) \geq 0. \tag{2.7}$$

If $\mathcal{A}$ has in addition a unit element 1 then ω is called a *state* if ω is positive and $\omega(1) = 1$. It follows that for every positive linear functional ω one has the Cauchy–Schwarz inequality (cf. [21, Lemma 5])

$$\omega(A^*B) = \overline{\omega(B^*A)}, \tag{2.8}$$

$$\omega(A^*B)\overline{\omega(A^*B)} \leq \omega(A^*A)\omega(B^*B). \tag{2.9}$$

Using the positive linear functionals we can define positivity for algebra elements as well. We have two reasonable possibilities for such a definition:

Definition 2.3. Let $\mathcal{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then a Hermitian element $A \in \mathcal{A}$ is called

1. algebraically positive if there exist elements $B_i \in \mathcal{A}$ and positive numbers $b_i \in \mathsf{R}$, where $i = 1, \ldots, n$ such that $A = b_1 B_1^* B_1 + \cdots + b_n B_n^* B_n$;
2. positive if for all positive linear functionals $\omega : \mathcal{A} \to \mathsf{C}$ one has $\omega(A) \geq 0$.

The set of algebraically positive elements is denoted by $\mathcal{A}^{++}$ and the set of positive elements is denoted by $\mathcal{A}^{+}$.

In principle there is still another possibility as we are dealing with rings: we call A *weakly algebraically positive* if there is a positive $p \in \mathsf{R}$ such that pA is algebraically positive. But this coincides with algebraic positivity as soon as we pass to the quotient fields. Clearly an algebraically positive element is positive whence $\mathcal{A}^{++} \subseteq \mathcal{A}^{+}$, but the converse is not true in general. Nevertheless, in a C^*-algebra over $\mathbb{C}$, both notions are known to coincide since here any positive element has a unique positive square root. As a first example of a *-algebra over C and the corresponding positive elements we mention the $n \times n$-matrices $M_n(\mathsf{C})$ as discussed in Appendix A. Moreover, we show in Appendix B that this definition yields the expected result for smooth functions on a manifold.

As in C^*-algebra theory, we use the positive elements $\mathcal{A}^{+}$ to endow the Hermitian elements with the structure of a half ordering by defining $A \geq B$ if $A - B \in \mathcal{A}^{+}$, where A, B are Hermitian. In addition, we have the following characterization of $\mathcal{A}^{++}$ and $\mathcal{A}^{+}$ analogously to the well-known case of C^*-algebras, see, e.g., [22].

Lemma 2.4. *Let $\mathcal{A}$ be a *-algebra over C. Then $\mathcal{A}^{++}$ and $\mathcal{A}^{+}$ are convex cones, i.e. for $A, B \in \mathcal{A}^{++}$ (resp. $\mathcal{A}^{+}$) and $a, b \geq 0$ we have $aA + bB \in \mathcal{A}^{++}$ (resp. $\mathcal{A}^{+}$). Furthermore, for any positive linear functional ω and any $C \in \mathcal{A}$ the functional $\omega_C : A \mapsto \omega(C^*AC)$ is positive and thus $C^*\mathcal{A}^{++}C \subseteq \mathcal{A}^{++}$ as well as $C^*\mathcal{A}^{+}C \subseteq \mathcal{A}^{+}$.*

Let us now introduce the notion of an approximate identity motivated by the usual C^*-algebra theory. Consider a directed set I, i.e. a partially ordered set I such that for each $\alpha, \beta \in I$ there exists a $\gamma \in I$ such that $\gamma \geq \alpha, \beta$. As we have no a priori notion of convergence we have to rely on the following algebraic definition. Let $\{E_\alpha\}_{\alpha \in I}$ be a set of elements $E_\alpha = E_\alpha^* \in \mathcal{A}$ such that for all $\alpha < \beta$ we have $E_\alpha = E_\alpha E_\beta = E_\beta E_\alpha$. Moreover, let $\mathcal{A}$ be filtered by subspaces $\mathcal{A}_\alpha$ also indexed by I, i.e. for all $\alpha \leq \beta$ one has $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ and $\mathcal{A} = \bigcup_{\alpha \in I} \mathcal{A}_\alpha$. Finally, assume that for all $A \in \mathcal{A}_\alpha$ one has $A = E_\alpha A = AE_\alpha$. In this case $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha \in I}$ is called an *approximate identity* for $\mathcal{A}$. Note that we do not require $E_\alpha^2 = E_\alpha$ nor $E_\alpha \in \mathcal{A}_\alpha$. In the following, we mainly consider *-algebras which admit such an approximate identity. If $\mathcal{A}$ has a unit element then clearly $\{\mathcal{A}, 1\}$ is an approximate identity. A less trivial example, and our main motivation, is given by $C_0^\infty(M)$, the algebra of complex-valued functions with compact support on a non-compact manifold (see Section 8). Using the Cauchy–Schwarz inequality one easily obtains the following lemma:

Lemma 2.5. *Let $\mathcal{A}$ be a *-algebra over* C *with approximate identity* $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha\in I}$ *and let* $\omega : \mathcal{A} \to \mathsf{C}$ *be a positive linear functional. Then* ω *is real, i.e.* $\omega(A^*) = \overline{\omega(A)}$ *for all* $A \in \mathcal{A}$. *Moreover, if for some* $\alpha \in I$ *one has* $\omega(E_\alpha^2) = 0$ *then* $\omega|_{\mathcal{A}_\alpha} = 0$.

Let us now discuss some notions concerning *-representations of a *-algebra over C. From Lemma 2.2, we observe that for a pre-Hilbert space $\mathfrak{H}$ over C, the algebra $\mathfrak{B}(\mathfrak{H})$ is also a *-algebra since any element $A \in \mathfrak{B}(\mathfrak{H})$ has a *unique* adjoint A^* and the map $A \mapsto A^*$ is obviously a *-involution. Then a *-*representation* π of a *-algebra $\mathcal{A}$ on $\mathfrak{H}$ is a *-homomorphism $\pi : \mathcal{A} \to \mathfrak{B}(\mathfrak{H})$, i.e. a linear map such that $\pi(AB) = \pi(A)\pi(B)$ and $\pi(A^*) = \pi(A)^*$. As usual, π is called *faithful* if π is injective, and *non-degenerate* if $\pi(A)\phi = 0$ for all A implies $\phi = 0$. It follows that if $\mathcal{A}$ has a unit element then π is non-degenerate if and only if $\pi(1) = \mathrm{id}$. We shall also make use of the following definition: a *-representation π is called *strongly non-degenerate* if the C-linear span of all vectors of the form $\pi(A)\phi$ with $A \in \mathcal{A}$ and $\phi \in \mathfrak{H}$ coincides with the whole space $\mathfrak{H}$. If $\mathcal{A}$ has a unit element, then clearly non-degeneracy and strong non-degeneracy are equivalent. In general, strong non-degeneracy implies non-degeneracy since if $(\pi, \mathfrak{H})$ is strongly non-degenerate and $\phi \in \mathfrak{H}$ is a vector such that $\pi(A)\phi = 0$ for all $A \in \mathcal{A}$ then $\langle\psi, \pi(A)\phi\rangle = \langle\pi(A^*)\psi, \phi\rangle = 0$ for all $\psi \in \mathfrak{H}$. But then we can chose A_i and ψ_i such that $\sum_i \pi(A_i^*)\psi_i = \phi$ due to the strong non-degeneracy, whence $\phi = 0$ follows. Thus $(\pi, \mathfrak{H})$ is also non-degenerate. Note that in the case of a *-representation of a C^*-algebra non-degeneracy implies that the span of all $\pi(A)\phi$ is dense in the Hilbert space $\mathfrak{H}$. In the following, the strongly non-degenerate case will be the most important one. Moreover, π is called *cyclic* with *cyclic vector* $\Omega \in \mathfrak{H}$ if for all $\psi \in \mathfrak{H}$ there is a $A \in \mathcal{A}$ such that $\psi = \pi(A)\Omega$. If any non-zero vector in $\mathfrak{H}$ is cyclic then π is called *transitive*. The pre-Hilbert space $\mathfrak{H}$ is called *filtered* if there is a directed set I and subspaces $\{\mathfrak{H}_\alpha\}_{\alpha\in I}$ of $\mathfrak{H}$ such that $\mathfrak{H}_\alpha \subseteq \mathfrak{H}_\beta$ for $\alpha \leq \beta$ and $\mathfrak{H} = \bigcup_{\alpha\in I}\mathfrak{H}_\alpha$. Then the representation π is called *compatible with the filtration* if $\pi(\mathcal{A})\mathfrak{H}_\alpha \subseteq \mathfrak{H}_\alpha$ for all $\alpha \in I$. Finally, we call π *pseudo-cyclic* if $\mathfrak{H}$ is filtered and each subspace $\mathfrak{H}_\alpha$ of the filtration is cyclic for π with cyclic vector Ω_α. In this case $\{\Omega_\alpha\}_{\alpha\in I}$ are called the *pseudo-cyclic vectors* of π. Note that π is not assumed to be compatible with the filtration. If one has *-representations $\pi^{(i)}$ for $i \in I$ on pre-Hilbert spaces $\mathfrak{H}^{(i)}$ then they induce a *-representation π on the direct orthogonal sum $\mathfrak{H} = \oplus_{i\in I}\mathfrak{H}^{(i)}$ in the usual way. If π has no non-trivial invariant subspace then π is called *irreducible*. If π is a direct orthogonal sum of pseudo-cyclic *-representations of $\mathcal{A}$ then π is clearly strongly non-degenerate.

Our main motivation to consider pseudo-cyclic representations is the fact that $C_0^\infty(M)$ acts in a pseudo-cyclic way on itself (by left-multiplication) but there is no cyclic vector if M is non-compact.

Let π_1 and π_2 be two *-representations of $\mathcal{A}$ on $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and let $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a linear map. Then T is called an *intertwiner* if $\pi_2(A)T = T\pi_1(A)$ for all $A \in \mathcal{A}$. We are mainly interested in *isometric*, *adjointable*, or *unitary* intertwiners. If π_1

is a *-representation of $\mathcal{A}$ on $\mathfrak{H}_1$ and T is a unitary map $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ then $\pi_2(A) := T\pi_1(A)T^{-1}$ defines a *-representation on $\mathfrak{H}_2$ and if for two *-representations there exists such a unitary intertwiner then these representations are called *unitarily equivalent*.

To study the *-representations of *-algebras, we consider the following categories. Denote by *-$\mathsf{Rep}(\mathcal{A})$ the category of *-representations of $\mathcal{A}$ on pre-Hilbert spaces over C with isometric (or adjointable, or unitary) intertwiners as morphisms. Since we shall mainly be interested in strongly non-degenerate *-representations of *-algebras, we denote by *-$\mathsf{Rep}(\mathcal{A})$ the category of strongly non-degenerate *-representations of $\mathcal{A}$.

Let us now briefly recall the algebraic GNS construction of *-representations using positive functionals, as discussed in detail in [20,21]. If $\omega : \mathcal{A} \to \mathsf{C}$ is a positive linear functional, the space $\mathcal{J}_\omega$ defined by

$$\mathcal{J}_\omega := \{A \in \mathcal{A} | \omega(A^*A) = 0\} \subseteq \mathcal{A} \tag{2.10}$$

is a left ideal called the *Gel'fand ideal* of ω. The quotient $\mathfrak{H}_\omega := \mathcal{A}/\mathcal{J}_\omega$ carries a Hermitian product defined by $\langle \psi_A, \psi_B \rangle := \omega(A^*B)$, where $\psi_A, \psi_B \in \mathfrak{H}_\omega$ denote the equivalence classes of $A, B \in \mathcal{A}$, respectively. Since $\mathcal{J}_\omega$ is a left ideal, $\mathfrak{H}_\omega$ is a $\mathcal{A}$-left module which gives rise to the *GNS representation* π_ω defined by $\pi_\omega(A)\psi_B := \psi_{AB}$. A straightforward computation shows $\pi_\omega(A) \in \mathfrak{B}(\mathfrak{H}_\omega)$ and $\pi_\omega(A^*) = \pi_\omega(A)^*$ whence π_ω is a *-representation.

Now assume in addition that $\mathcal{A}$ has an approximate identity $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha\in I}$ and define $\mathfrak{H}_{\omega,\alpha} := \pi_\omega(\mathcal{A})\psi_{E_\alpha}$ for $\alpha \in I$. By definition $\mathfrak{H}_{\omega,\alpha}$ is a subspace of $\mathfrak{H}_\omega$ which is cyclic for π_ω with cyclic vector ψ_{E_α}, though it may happen that $\mathfrak{H}_{\omega,\alpha} = \{0\}$ for some α. Moreover, one immediately verifies that $\mathfrak{H}_{\omega,\alpha} \subseteq \mathfrak{H}_{\omega,\beta}$ for $\alpha \le \beta$ and $\mathfrak{H}_\omega = \bigcup_{\alpha\in I} \mathfrak{H}_{\omega,\alpha}$. Thus π_ω is pseudo-cyclic with pseudo-cyclic vectors $\{\psi_{E_\alpha}\}_{\alpha\in I}$. Finally, note that π_ω is compatible with this filtration and clearly

$$\langle \psi_{E_\alpha}, \pi_\omega(A)\psi_{E_\alpha} \rangle = \omega(E_\alpha A E_\alpha) \tag{2.11}$$

for all $\alpha \in I$ and $A \in \mathcal{A}$. On the other hand, the GNS representation is already characterized by this property.

Lemma 2.6. *Let $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha\in I}$ be an approximate identity of $\mathcal{A}$ and $\omega : \mathcal{A} \to \mathsf{C}$ a positive linear functional. If π is a pseudo-cyclic *-representation on $\mathfrak{H} = \bigcup_{\alpha\in I} \mathfrak{H}_\alpha$ with pseudo-cyclic vectors Ω_α* (*same index set, but some Ω_α may be zero*) *which is compatible with the filtration such that $\langle \Omega_\alpha, \pi(A)\Omega_\alpha \rangle = \omega(E_\alpha A E_\alpha)$ for all $\alpha \in I$ and $A \in \mathcal{A}$, then π is unitarily equivalent to the GNS representation π_ω by the filtration preserving unitary map*

$$U : \mathfrak{H}_{\omega,\alpha} \ni \pi_\omega(A)\psi_{E_\alpha} \mapsto \pi(A)\Omega_\alpha \in \mathfrak{H}_\alpha. \tag{2.12}$$

The proof is a straightforward verification using only the definitions. Note that in particular U maps ψ_{E_α} to Ω_α. Note also that a GNS representation of a *-algebra which has an approximate identity is always strongly non-degenerate and thus non-degenerate. This is a main reason why we are interested in *-$\mathsf{Rep}(\mathcal{A})$.

The following additional property of a *-algebra provides some C^*-algebra-like features concerning Hermitian elements and faithful *-representations.

Definition 2.7. Let $\mathcal{A}$ be a *-algebra over C. Then $\mathcal{A}$ has sufficiently many positive linear functionals if for any non-zero Hermitian element H there exists a positive linear functional of $\mathcal{A}$ such that $\omega(H) \neq 0$.

Proposition 2.8. *Let $\mathcal{A}$ be a *-algebra over C with an approximate identity. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has sufficiently many positive linear functionals.*
2. *For any non-zero Hermitian element $H \in \mathcal{A}$ there exists a *-representation π of $\mathcal{A}$ with $\pi(H) \neq 0$.*
3. *There exists a faithful *-representation of $\mathcal{A}$.*

   In this case the following properties are also fulfilled:
4. *If for $A \in \mathcal{A}$ one has $A^*A = 0$ then $A = 0$.*
5. *There are no non-zero nilpotent normal elements in $\mathcal{A}$.*
6. *$\mathcal{A}$ is torsion-free, i.e. $zA = 0$ for $0 \neq z \in \mathsf{C}$ and $A \in \mathcal{A}$ implies $A = 0$.*

Proof. Assume (1) and let $0 \neq H \in \mathcal{A}$ be Hermitian and let $\alpha \in I$ be an index of the approximate identity such that $HE_\alpha = H = E_\alpha H$ and choose a positive linear functional ω with $\omega(H) \neq 0$. Then $\omega(H) = \omega(E_\alpha H E_\alpha) = \langle \psi_{E_\alpha}, \pi_\omega(H)\psi_{E_\alpha}\rangle_\omega$ shows that $\pi_\omega(H) \neq 0$ in the GNS representation corresponding to ω proving (2). Assume (2), then the orthogonal sum over all GNS representations π is faithful: it is clear that $\pi(H) \neq 0$ for $H \neq 0$ if H is Hermitian or anti-Hermitian. Let $A \neq 0$ be not anti-Hermitian. Then $A + A^* \neq 0$ and thus $\pi(A + A^*) \neq 0$ since $A + A^*$ is Hermitian. Thus $\pi(A) = \pi(A^*)^*$ is also non-zero proving (3). Finally, let π be a faithful *-representation. Thus it is sufficient to prove (1) for (a *-subalgebra of) $\mathfrak{B}(\mathfrak{H})$ for an arbitrary pre-Hilbert space $\mathfrak{H}$. Let $H = H^* \in \mathfrak{B}(\mathfrak{H})$ be such that for all $\psi \in \mathfrak{H}$ we have $\langle \psi, H\psi\rangle = 0$. Then by the usual polarization argument and the fact that $2 \neq 0$ in R we conclude $\langle \psi, H\phi\rangle = 0$ for all $\psi, \phi \in \mathfrak{H}$. Hence, by the non-degeneracy of the Hermitian product, $H = 0$ follows. Thus for a non-zero Hermitian $H \in \mathfrak{B}(\mathfrak{H})$ there exists a vector $\psi \in \mathfrak{H}$ with $\langle \psi, H\psi\rangle \neq 0$. Then $A \mapsto \langle \psi, A\psi\rangle$ is the desired positive linear functional proving the equivalence of the first three properties. Now assume that they are fulfilled. Then (4) follows immediately from the fact that one has a faithful *-representation. Now let $H \neq 0$ be Hermitian and E_α as above and ω a positive linear functional with $\omega(H) \neq 0$. By the Cauchy–Schwarz inequality we have $\omega(H)\overline{\omega(H)} \leq \omega(E_\alpha^2)\omega(H^2)$ whence $\omega(H^2) \neq 0$. By induction we conclude that $H^{2^n} \neq 0$ and thus H cannot be nilpotent. This proves (5) for Hermitian elements. Together with (4), it also follows for normal elements. Finally, for (6) pass to a faithful *-representation and take expectation values. □

Corollary 2.9. *Let $\mathcal{A}$ be a *-algebra over C with sufficiently many positive linear functionals and approximate identity, and let $A \in \mathcal{A}$. If $\omega(A) = 0$ for all positive linear functionals then $A = 0$.*

Proof. This follows since $2A$ can be written as complex linear combination of the Hermitian elements $A + A^*$ and $\mathrm{i}(A - A^*)$. □

We shall see examples for *-algebras with sufficiently many positive linear functionals later in this work and refer also to the (counter-)examples in [23, Section 2].

3. Bimodules and algebraic Rieffel induction

Now we want to transfer the usual construction of induced representations using Rieffel induction (see [65] and, e.g., the textbook [51]) from the setting of C^*-algebras to the more algebraic framework of *-algebras over ordered rings.

Let $\mathcal{A}$, $\mathcal{B}$ be two *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, where R is an ordered ring. Then we consider a ($\mathcal{B}$-$\mathcal{A}$)-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, i.e. a C-module endowed with a $\mathcal{B}$-left action $\mathsf{L}_{\mathcal{B}}$ and a $\mathcal{A}$-right action $\mathsf{R}_{\mathcal{A}}$ written as

$$\mathsf{L}_{\mathcal{B}} : \mathcal{B} \to \mathrm{End}_{\mathsf{C}}({}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}) \leftarrow \mathcal{A} : \mathsf{R}_{\mathcal{A}}, \tag{3.1}$$

such that the left action with elements in $\mathcal{B}$ and the right action with elements in $\mathcal{A}$ commute. We sometimes omit the explicit use of the maps $\mathsf{L}_{\mathcal{B}}$ and $\mathsf{R}_{\mathcal{A}}$ and simply write $B \cdot x$ and $x \cdot A$, respectively, where $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$.

As an additional structure we consider a positive semi-definite $\mathcal{A}$-valued inner product (a 'rigging map') on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ which is a map

$$\langle \cdot, \cdot \rangle_{\mathcal{A}} : {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \times {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \to \mathcal{A}, \tag{3.2}$$

satisfying the following defining properties

(X1) $\langle x, ay + bz \rangle_{\mathcal{A}} = a\langle x, y \rangle_{\mathcal{A}} + b\langle x, z \rangle_{\mathcal{A}}$,
(X2) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$,
(X3) $\langle x, y \cdot A \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}} A$,
(X4) $\langle x, x \rangle_{\mathcal{A}} \geq 0$,

for all $a, b \in \mathsf{C}$, $A \in \mathcal{A}$ and $x, y, z \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. The positivity requirement can be sharpened in two directions: we consider the following algebraic positivity

(X4a) $\langle x, x \rangle_{\mathcal{A}} \in \mathcal{A}^{++}$,

and the positive definiteness conditions

(X4′) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ implies $x = 0$,
(X4a′) $\langle x, x \rangle_{\mathcal{A}} \in \mathcal{A}^{++}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ implies $x = 0$,

for all $x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. For most of our applications, (X4) will turn out to be sufficient and clearly (X4a′) implies (X4a) as well as (X4′), and (X4a) as well as (X4′) imply (X4). Besides these axioms for the $\mathcal{A}$-valued inner product, we shall need a compatibility of the inner product with the $\mathcal{B}$-left action on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ which motivates the requirement

(X5) $\langle x, B \cdot y \rangle_{\mathcal{A}} = \langle B^* \cdot x, y \rangle_{\mathcal{A}}$

for all $x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and $B \in \mathcal{B}$. For later use, we shall also mention the following *fullness condition*:

(X6) $\mathcal{A} = \mathsf{C}\text{-span}\, \{\langle x, y \rangle_{\mathcal{A}} | x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}\}$, which will guarantee the non-triviality of the constructions that follow. In the usual C^*-algebra approach, one only demands that the span of all inner products $\langle x, y\rangle_{\mathcal{A}}$ is dense in $\mathcal{A}$ but as we do not have any topologies we have to demand (X6).

Now we have all the requisites to describe the algebraic Rieffel induction following almost literally the construction known from C^*-algebras. We start with a *-representation $\pi_{\mathcal{A}}$ of $\mathcal{A}$ on $\mathfrak{H}$ and assume we have a bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfying the axioms (X1)–(X5). Then we shall construct a *-representation of $\mathcal{B}$. To this end we consider the C-module

$$\tilde{\mathfrak{K}} := {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}, \tag{3.3}$$

where the '$\mathcal{A}$-balanced' tensor product $\otimes_{\mathcal{A}}$ is defined by using the right action of $\mathcal{A}$ on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and the left representation $\pi_{\mathcal{A}}$ on $\mathfrak{H}$, i.e. we consider the tensor product ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathsf{C}} \mathfrak{H}$ and the subspace N spanned by elements of the form $x \cdot A \otimes \psi - x \otimes \pi_{\mathcal{A}}(A)\psi$. Then $\tilde{\mathfrak{K}} :={}_{\mathcal{B}} \mathfrak{X}_{\mathcal{A}} \otimes_{\mathsf{C}} \mathfrak{H}/N$. In other words we identify $x \cdot A \otimes \psi$ with $x \otimes \pi_{\mathcal{A}}(A)\psi$ for all $x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, $A \in \mathcal{A}$, and $\psi \in \mathfrak{H}$. Then $\tilde{\mathfrak{K}}$ carries a canonical $\mathcal{B}$-left action which we shall denote by $\tilde{\pi}_{\mathcal{B}}$ given by

$$\tilde{\pi}_{\mathcal{B}}(B)(x \otimes \psi) := (\mathsf{L}_{\mathcal{B}}(B)x) \otimes \psi = (B \cdot x) \otimes \psi. \tag{3.4}$$

Note that this is indeed well-defined on $\tilde{\mathfrak{K}}$ since $\mathsf{L}_{\mathcal{B}}$ and $\mathsf{R}_{\mathcal{A}}$ commute. Moreover, since $\mathsf{L}_{\mathcal{B}}$ is a $\mathcal{B}$-representation it follows that $\tilde{\pi}_{\mathcal{B}}$ is also a $\mathcal{B}$-representation. Next we want to equip $\tilde{\mathfrak{K}}$ with the structure of a positive semi-definite Hermitian product. Following the usual construction we define for $x \otimes \psi$, $y \otimes \phi \in \tilde{\mathfrak{K}}$

$$\langle x \otimes \psi, y \otimes \phi\rangle_{\tilde{\mathfrak{K}}} := \langle \psi, \pi_{\mathcal{A}}(\langle x, y\rangle_{\mathcal{A}})\phi\rangle_{\tilde{\mathfrak{H}}}, \tag{3.5}$$

and extend this by linearity in the second and antilinearity in the first argument to an inner product on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathsf{C}} \mathfrak{H}$. A simple computation shows that $\langle\cdot,\cdot\rangle_{\tilde{\mathfrak{K}}}$ is indeed well-defined on $\tilde{\mathfrak{K}}$. Moreover, the inner product $\langle\cdot,\cdot\rangle_{\tilde{\mathfrak{K}}}$ enjoys the symmetry property

$$\langle x \otimes \psi, y \otimes \phi\rangle_{\tilde{\mathfrak{K}}} = \overline{\langle y \otimes \phi, x \otimes \psi\rangle},$$

as an easy computation shows. Next we consider the compatibility of $\langle\cdot,\cdot\rangle_{\tilde{\mathfrak{K}}}$ with $\tilde{\pi}_{\mathcal{B}}$. Let $x \otimes \psi$, $y \otimes \phi \in \tilde{\mathfrak{K}}$ be elementary tensors and $B \in \mathcal{B}$. Then an easy computation using (X5) shows

$$\langle x \otimes \psi, \tilde{\pi}_{\mathcal{B}}(B) y \otimes \phi\rangle_{\tilde{\mathfrak{K}}} = \langle \tilde{\pi}_{\mathcal{B}}(B^*) x \otimes \psi, y \otimes \phi\rangle_{\tilde{\mathfrak{K}}}. \tag{3.6}$$

By linearity it follows that $\tilde{\pi}_{\mathcal{B}}(B^*)$ is an adjoint of $\tilde{\pi}_{\mathcal{B}}(B)$.

It remains to prove that $\langle\cdot,\cdot\rangle_{\tilde{\mathfrak{K}}}$ is positive semi-definite. First notice that for all $\psi \in \mathfrak{H}$ the linear functional $A \mapsto \langle \psi, \pi_{\mathcal{A}}(A)\psi\rangle_{\mathfrak{H}}$ is a positive linear functional on $\mathcal{A}$. Thus we obtain for elementary tensors $x \otimes \psi \in \tilde{\mathfrak{K}}$

$$\langle x \otimes \psi, x \otimes \psi\rangle_{\tilde{\mathfrak{K}}} = \langle \psi, \pi_{\mathcal{A}}(\langle x, x\rangle_{\mathcal{A}})\psi\rangle_{\mathfrak{H}} \geq 0, \tag{3.7}$$

since by (X4) the algebra element $\langle x, x\rangle_{\mathcal{A}} \in \mathcal{A}$ is positive. Note that our definition of positive algebra elements comes in crucially in this context. Though $\langle\cdot,\cdot\rangle_{\tilde{\mathfrak{K}}}$ is positive on the elementary tensors in $\tilde{\mathfrak{K}}$, we cannot a priori guarantee the positivity for arbitrary elements of the form

$x_1 \otimes \psi_1 + \cdots + x_n \otimes \psi_n \in \tilde{\mathfrak{K}}$. In the case of C^*-algebras one uses the fact that a non-degenerate *-representation π is the direct orthogonal sum of cyclic representations. Thus any element in $\tilde{\mathfrak{K}}$ can be written as an orthogonal sum of elementary tensors and the positivity is easily established, see, e.g., [51, Chapter VI, Section 2.2] or [64, Proposition 2.64].

As *-representations in our setting might not satisfy this condition in general, we have to impose additional properties of the bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ which are sufficient to guarantee the positivity of $\langle \cdot, \cdot \rangle_{\tilde{\mathfrak{K}}}$. We define the following property:

(P) The inner product $\langle \cdot, \cdot \rangle_{\tilde{\mathfrak{K}}}$ is positive semi-definite for all representations $(\mathfrak{H}, \pi_{\mathcal{A}})$ of $\mathcal{A}$.

We list conditions which will imply this property, but remark that there are situations where the positivity can be proven by other methods, see, e.g., the next section.

(P1) ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = \oplus_{i \in I} \mathfrak{X}^{(i)}$ and $\mathfrak{X}^{(i)} \perp \mathfrak{X}^{(j)}$ for all $i \neq j \in I$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A}}$.
(P2) The $\mathcal{A}$-right action $\mathsf{R}_{\mathcal{A}}$ preserves this direct sum.
(P3) Each $\mathfrak{X}^{(i)}$ is pseudo-cyclic for $\mathsf{R}_{\mathcal{A}}$ with directed filtered submodules $\mathfrak{X}^{(i)} = \bigcup_{\alpha \in I^{(i)}} \mathfrak{X}^{(i)}_{\alpha}$ and pseudo-cyclic vectors $\Omega^{(i)}_{\alpha}$.

We also define a slightly weaker form of pseudo-cyclicity for the bimodule:

(PC) ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = \sum_{i \in I} \mathfrak{X}^{(i)}$ with orthogonal C-submodules $\mathfrak{X}^{(i)}$ for $i \neq j$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ such that each $\mathfrak{X}^{(i)}$ is pseudo-cyclic for $\mathsf{R}_{\mathcal{A}}$ with directed filtered submodules $\mathfrak{X}^{(i)} = \bigcup_{\alpha \in I^{(i)}} \mathfrak{X}^{(i)}$ and pseudo-cyclic vectors $\Omega^{(i)}_{\alpha}$.

Note that for (PC) we do not require the sum decomposition to be direct since $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ may be degenerate and moreover, we do not require the sum decomposition or the filtrations to be compatible with $\mathsf{R}_{\mathcal{A}}$.

Lemma 3.1. (P1)–(P3) $\Rightarrow$ (PC) $\Rightarrow$ (P).

Proof. The first implication is obvious so let us prove the second. Let $(\mathfrak{H}, \pi_{\mathcal{A}})$ be a *-representation of $\mathcal{A}$ and consider $\tilde{\mathfrak{K}} = {}_{\mathcal{B}}\, \mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}$. Define $\tilde{\mathfrak{K}}^{(i)} = \mathfrak{X}^{(i)} \otimes_{\mathcal{A}} \mathfrak{H}$ which is a C-submodule of $\tilde{\mathfrak{K}}$ for all $i \in I$ and clearly $\sum_{i \in I} \tilde{\mathfrak{K}}^{(i)} = \tilde{\mathfrak{K}}$ though the sum may not be direct. Even if the sum decomposition of ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ were direct, the identifications in the $\mathcal{A}$-balanced tensor product could make the sum decomposition of $\tilde{\mathfrak{K}}$ non-direct. Nevertheless they are orthogonal as one immediately can verify using (PC). To show that $\langle \cdot, \cdot \rangle_{\tilde{\mathfrak{K}}}$ is positive semi-definite we may restrict to $\tilde{\mathfrak{K}}^{(i)}$ for fixed i due to their orthogonality. Let $\chi^{(i)} = x^{(i)}_1 \otimes \phi_1 + \cdots + x^{(i)}_n \otimes \phi_n$ with $x^{(i)}_k \in \mathfrak{X}^{(i)}$ and $\phi_k \in \mathfrak{H}$ for $k = 1, \ldots, n$. Then there is a $\alpha \in I^{(i)}$ such that $x^{(i)}_1, \ldots, x^{(i)}_n \in \mathfrak{X}^{(i)}_{\alpha}$ and hence we find $A_1, \ldots, A_n \in \mathcal{A}$ such that $x^{(i)}_k = \Omega^{(i)}_{\alpha} \cdot A_k$ for $k = 1, \ldots, n$. Thus $\chi^{(i)} = \Omega^{(i)}_{\alpha} \otimes \phi$ with $\phi = \pi_{\mathcal{A}}(A_1)\phi_1 + \cdots + \pi_{\mathcal{A}}(A_n)\phi_n$. Hence by (3.7) the positivity $\langle \chi^{(i)}, \chi^{(i)} \rangle_{\tilde{\mathfrak{K}}} \geq 0$ easily follows, proving (P). □

Nevertheless, in most of our examples we shall deal with (P1)–(P3) and not with (PC). We shall even encounter situations where we can prove (P) directly without (P1)–(P3) or (PC). Taking (P1)–(P3) or (PC) as an example of how to guarantee (P), we investigate now the consequences of (P) in general. The following technical remark will be useful in a few situations.

Lemma 3.2. *Assume ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies* (P) *and let $\tilde{\mathfrak{H}}$ be a* C*-module with positive semi-definite Hermitian product and a representation of $\mathcal{A}$ by adjointable operators. Then the inner product defined by* (3.5) *on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \tilde{\mathfrak{H}}$ is positive semi-definite.*

Proof. This is a simple consequence of (P) obtained by passing to the pre-Hilbert space $\tilde{\mathfrak{H}}/\tilde{\mathfrak{H}}^{\perp}$. □

Under the assumption that ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies (P) we obtain a positive semi-definite Hermitian product for $\tilde{\mathfrak{K}}$. Moreover, $\tilde{\pi}_{\mathcal{B}}(B) \in \mathfrak{B}(\tilde{\mathfrak{K}})$ due to (3.6) for all $B \in \mathcal{B}$. Nevertheless the inner product $\langle \cdot, \cdot \rangle_{\tilde{\mathfrak{K}}}$ may be degenerate and thus we have to quotient out the vectors of length zero. Hence we define

$$\mathfrak{K} := \tilde{\mathfrak{K}}/\tilde{\mathfrak{K}}^{\perp}, \tag{3.8}$$

which is now a pre-Hilbert space over C. The following simple lemma ensures that we obtain a *-representation of $\mathcal{B}$ on $\tilde{\mathfrak{K}}$:

Lemma 3.3. *Let $\tilde{\mathfrak{H}}$ be a* C*-module with semi-definite Hermitian product and let $\mathfrak{H} = \tilde{\mathfrak{H}}/\tilde{\mathfrak{H}}^{\perp}$.*

1. *The algebra $\mathfrak{B}(\tilde{\mathfrak{H}})/\mathfrak{I}(\tilde{\mathfrak{H}})$ has a canonical *-involution given by $[A]^* := [A^*]$, where A^* is an adjoint of $A \in \mathfrak{B}(\tilde{\mathfrak{H}})$.*
2. *The map $\mathfrak{B}(\tilde{\mathfrak{H}})/\mathfrak{I}(\tilde{\mathfrak{H}}) \ni [A] \mapsto ([\psi] \mapsto [A\psi] \in \mathfrak{H}) \in \mathfrak{B}(\mathfrak{H})$ is an injective *-homomorphism.*

From this lemma and (3.6) we conclude that the representation $\tilde{\pi}_{\mathcal{B}}$ of $\mathcal{B}$ on $\tilde{\mathfrak{K}}$ passes to the quotient $\mathfrak{K}$ and yields a *-representation $\pi_{\mathcal{B}}$ of $\mathcal{B}$ on $\mathfrak{K}$ given on elementary tensors by

$$\pi_{\mathcal{B}}(B)[x \otimes \psi] := [\tilde{\pi}_{\mathcal{B}}(B)(x \otimes \psi)] = [B \cdot x \otimes \psi] \tag{3.9}$$

for $B \in \mathcal{B}$ and $x \otimes \psi \in \tilde{\mathfrak{K}}$. We shall call $\pi_{\mathcal{B}}$ the *induced representation* of $\mathcal{B}$ and the above construction shall be called the *algebraic Rieffel induction* in analogy to the Rieffel induction in the theory of C^*-algebras.

Proposition 3.4. *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over* C *and ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ a ($\mathcal{B}$-$\mathcal{A}$)-bimodule satisfying* (X1)–(X5) *and* (P). *Then for any *-representation $\pi_{\mathcal{A}}$ on a pre-Hilbert space $\mathfrak{H}$ the space $\tilde{\mathfrak{K}} = {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}$ carries a $\mathcal{B}$-representation $\tilde{\pi}_{\mathcal{B}}$ and a positive semi-definite Hermitian product which induce a *-representation $\pi_{\mathcal{B}}$ of $\mathcal{B}$ on the pre-Hilbert space $\mathfrak{K} := \tilde{\mathfrak{K}}/\tilde{\mathfrak{K}}^{\perp}$.*

To complete the construction of induced representations we have to check whether the above construction is functorial. This can be done as in the C^*-algebra case. Let $(\mathfrak{H}_1, \pi_{\mathcal{A}}^{(1)})$ and $(\mathfrak{H}_2, \pi_{\mathcal{A}}^{(2)})$ be two *-representations of $\mathcal{A}$ and let $U : \mathfrak{H}_1 \to \mathfrak{H}_2$ be an intertwiner. Then we define $\tilde{V} : \tilde{\mathfrak{K}}_1 \to \tilde{\mathfrak{K}}_2$ by

$$\tilde{V}(x \otimes \psi) := x \otimes U\psi \tag{3.10}$$

for $x \otimes \psi \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}_1$ and extend this by linearity. First note that $\tilde{V}$ is indeed well-defined since U is an intertwiner. Moreover, we clearly have for all $B \in \mathcal{B}$

$$\tilde{V}\left(\tilde{\pi}^{(1)}_{\mathcal{B}}(B)(x\otimes\psi)\right)=\tilde{\pi}^{(2)}_{\mathcal{B}}(B)\left(\tilde{V}(x\otimes\psi)\right), \tag{3.11}$$

whence $\tilde{V}$ is an intertwiner from $\tilde{\pi}^{(1)}_{\mathcal{B}}$ to $\tilde{\pi}^{(2)}_{\mathcal{B}}$. If we assume in addition that U is an isometric intertwiner, then a simple computation shows that $\tilde{V}:\tilde{\mathfrak{K}}_1\to\tilde{\mathfrak{K}}_2$ is also isometric. Thus $\tilde{V}$ passes to the quotients and yields an isometric map $V:\mathfrak{K}_1\to\mathfrak{K}_2$ which now is an isometric intertwiner from $\pi^{(1)}_{\mathcal{B}}$ to $\pi^{(2)}_{\mathcal{B}}$. Analogously, if U is an intertwiner with adjoint, then $\tilde{V}$ also has an adjoint and passes to the quotient as an adjointable V. We conclude that the algebraic Rieffel induction is indeed functorial in the category of *-representations with isometric or adjointable intertwiners. Moreover, we emphasize that if U is unitary then V is unitary as well. For a given bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ which satisfies (X1)–(X5) and (P) we denote the corresponding *functor* by $\mathfrak{R}_{\mathfrak{X}}:{}^*\text{-rep}(\mathcal{A})\to{}^*\text{-rep}(\mathcal{B})$, where $\mathfrak{R}_{\mathfrak{X}}:(\mathfrak{H},\pi_{\mathcal{A}})\mapsto(\mathfrak{R}_{\mathfrak{X}}\mathfrak{H}:=\mathfrak{K},\mathfrak{R}_{\mathfrak{X}}\pi_{\mathcal{A}}:=\pi_{\mathcal{B}})$ and $\mathfrak{R}_{\mathfrak{X}}:U\mapsto\mathfrak{R}_{\mathfrak{X}}U:=V$ as above.

Theorem 3.5. *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over* C. *Then any* $(\mathcal{B}$-$\mathcal{A})$*-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ which satisfies* (X1)–(X5) *and* (P) *yields a functor* $\mathfrak{R}_{\mathfrak{X}}:{}^*\text{-rep}(\mathcal{A})\to{}^*\text{-rep}(\mathcal{B})$.

Let us finally discuss the following non-degeneracy properties of the Rieffel induction: as for the case of *-representations we call the left-action of $\mathcal{B}$ on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *strongly non-degenerate* if the C-span of all $B\cdot x$ with $B\in\mathcal{B}$ and $x\in{}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ coincides with the whole space ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, and analogously for the $\mathcal{A}$-right action. Then a straightforward computation yields the following result:

Proposition 3.6. *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over* C *and ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ a bimodule satisfying* (X1)–(X5) *and* (P). *If in addition the $\mathcal{B}$-left action $\mathsf{L}_{\mathcal{B}}$ on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is strongly non-degenerate then the functor $\mathfrak{R}_{\mathfrak{X}}$ maps* $^*\text{-rep}(\mathcal{A})$ *into* $^*\text{-Rep}(\mathcal{B})$.

Remark 3.7. *If the bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies* (P1)–(P3) *then the right-action $\mathsf{R}_{\mathcal{A}}$ of $\mathcal{A}$ is automatically strongly non-degenerate*.

4. Properties of the algebraic Rieffel induction

This section shall be dedicated to some standard constructions and first results on the algebraic Rieffel induction, most of which have their analogues in the theory of C^*-algebras.

First we shall consider the behavior of $\mathfrak{R}_{\mathfrak{X}}$ with respect to direct sums of representations. Let $\{\mathfrak{H}^{(i)},\pi^{(i)}_{\mathcal{A}}\}_{i\in I}$ be *-representations of $\mathcal{A}$ and let $\mathfrak{H}:=\oplus_{i\in I}\mathfrak{H}^{(i)}$ be endowed with the *-representation $\pi_{\mathcal{A}}:=\oplus_{i\in I}\pi^{(i)}_{\mathcal{A}}$. Then canonically

$$\tilde{\mathfrak{K}}={}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}\otimes_{\mathcal{A}}\mathfrak{H}\cong\bigoplus_{i\in I}{}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}\otimes_{\mathcal{A}}\mathfrak{H}^{(i)}\cong\bigoplus_{i\in I}\tilde{\mathfrak{K}}^{(i)}, \tag{4.1}$$

since the representation $\pi_{\mathcal{A}}$ preserves the orthogonal sum decomposition of $\mathfrak{H}$. Moreover, $\tilde{\mathfrak{K}}^{(i)}$ and $\tilde{\mathfrak{K}}^{(j)}$ are orthogonal for $i\neq j$ whence (4.1) is an orthogonal decomposition of $\tilde{\mathfrak{K}}$. Note also that $\tilde{\pi}_{\mathcal{B}}$ preserves this direct sum and $\tilde{\pi}_{\mathcal{B}}|_{\tilde{\mathfrak{K}}^{(i)}}=\tilde{\pi}^{(i)}_{\mathcal{B}}$ for all $i\in I$ whence

$\tilde{\pi}_{\mathcal{B}} = \oplus_{i\in I}\tilde{\pi}_{\mathcal{B}}^{(i)}$. Finally, as this direct sum is orthogonal, the decompositions of $\tilde{\mathfrak{K}}$ and $\tilde{\pi}_{\mathcal{B}}$ induce a corresponding decomposition of $\mathfrak{K}$ and $\pi_{\mathcal{B}}$. Thus we have the following lemma:

Lemma 4.1. *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over* C *and* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *a bimodule satisfying* (X1)–(X5) *and* (P). *Then canonically*

$$\mathfrak{R}_{\mathfrak{X}}\left(\bigoplus_{i\in I}\mathfrak{H}^{(i)}, \bigoplus_{i\in I}\pi_{\mathcal{A}}^{(i)}\right) \cong \left(\bigoplus_{i\in I}\mathfrak{R}_{\mathfrak{X}}\mathfrak{H}^{(i)}, \bigoplus_{i\in I}\mathfrak{R}_{\mathfrak{X}}\pi_{\mathcal{A}}^{(i)}\right) \tag{4.2}$$

for any *-representations $\{\mathfrak{H}^{(i)}, \pi_{\mathcal{A}}^{(i)}\}_{i\in I}$ of $\mathcal{A}$.

Note that it still may happen that $\mathfrak{R}_{\mathfrak{X}}(\mathfrak{H}, \pi_{\mathcal{A}})$ is trivial since we have not yet imposed any non-triviality conditions on the bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ such as the fullness condition (X6) or the strong non-degeneracy of $\mathsf{L}_{\mathcal{B}}$. Nevertheless, the above lemma is very useful for questions of irreducibility of the induced representations.

On the other hand, it was argued in [75] that the question of whether a representation is irreducible or not is from the physical point of view in deformation quantization sometimes *not* the most important one, and the question of whether the *commutant* of the representation is trivial or not leads to physically more reasonable characterizations of the representations. Though both concepts are known to coincide in the case of C^*-algebras, this need not be true in the general case of *-algebras over ordered rings, see [75] for examples. Thus we consider for a *-representation $\pi_{\mathcal{A}}$ of $\mathcal{A}$ on $\mathfrak{H}$ the commutant

$$\pi_{\mathcal{A}}(\mathcal{A})' := \{C \in \mathfrak{B}(\mathfrak{H}) | \forall A \in \mathcal{A} : C\pi_{\mathcal{A}}(A) = \pi_{\mathcal{A}}(A)C\} \tag{4.3}$$

within $\mathfrak{B}(\mathfrak{H})$. Clearly $\pi_{\mathcal{A}}(\mathcal{A})'$ is a *-subalgebra of $\mathfrak{B}(\mathfrak{H})$ and we have $\pi_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{A}}(\mathcal{A})''$ and $\pi_{\mathcal{A}}(\mathcal{A})''' = \pi_{\mathcal{A}}(\mathcal{A})'$ as usual. Let $\mathcal{B}$ and ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be given as above and consider $C \in \pi_{\mathcal{A}}(\mathcal{A})'$. Then we define $\tilde{\Phi}_{\mathfrak{X}}(C) : \tilde{\mathfrak{K}} \to \tilde{\mathfrak{K}}$ by

$$\tilde{\Phi}_{\mathfrak{X}}(C)(x \otimes \psi) := x \otimes C\psi \tag{4.4}$$

for elementary tensors and extend this by linearity. Clearly $\tilde{\Phi}_{\mathfrak{X}}(C)$ is well-defined since $C \in \pi_{\mathcal{A}}(\mathcal{A})'$. Moreover, since $C \in \mathfrak{B}(\mathfrak{H})$ we have an adjoint C^* of C and thus it follows easily that $\tilde{\Phi}_{\mathfrak{X}}(C^*)$ is an adjoint of $\tilde{\Phi}_{\mathfrak{X}}(C)$. Thus we conclude that $\tilde{\Phi}_{\mathfrak{X}}(C) \in \mathfrak{B}(\tilde{\mathfrak{K}})$. By Lemma 3.3 it follows that $\tilde{\Phi}_{\mathfrak{X}}(C)$ and $\tilde{\Phi}_{\mathfrak{X}}(C^*)$ pass both to the quotient $\mathfrak{K}$ and yield $\Phi_{\mathfrak{X}}(C), \Phi_{\mathfrak{X}}(C^*) \in \mathfrak{B}(\mathfrak{K})$ which are adjoints of each other. Note finally that $\tilde{\Phi}_{\mathfrak{X}}(C)$ is clearly in the commutant of $\tilde{\pi}_{\mathcal{B}}(\mathcal{B})$ and thus $\Phi_{\mathfrak{X}}(C)$ is in the commutant of $\pi_{\mathcal{B}}(\mathcal{B})$. A last easy check shows that the map $C \mapsto \Phi_{\mathfrak{X}}(C)$ is a *-homomorphism of $\pi_{\mathcal{A}}(\mathcal{A})'$ into $\pi_{\mathcal{B}}(\mathcal{B})'$. We summarize the result in the following proposition:

Proposition 4.2. *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over* C *and* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *a bimodule satisfying* (X1)–(X5) *and* (P). *Then the functor $\mathfrak{R}_{\mathfrak{X}}$ yields a *-homomorphism $\Phi_{\mathfrak{X}} : \pi_{\mathcal{A}}(\mathcal{A})' \to ((\mathfrak{R}_{\mathfrak{X}}\pi_{\mathcal{A}})(\mathcal{B}))'$ for all *-representations* $(\mathfrak{H}, \pi_{\mathcal{A}})$.

Let us now investigate the relation between the algebraic Rieffel induction and tensor products of *-algebras. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are *-algebras over C then the tensor product

$\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$ (taken over C) is again an associative algebra over C, and by setting

$$(A_1 \otimes A_2)^* := A_1^* \otimes A_2^*, \tag{4.5}$$

we clearly obtain a *-involution for $\mathcal{A}$ whence $\mathcal{A}$ becomes a *-algebra over C.

Lemma 4.3. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be *-algebras over C and $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ their tensor product.*

1. *If $A_1 \in \mathcal{A}_1^{++}$ and $A_2 \in \mathcal{A}_2^{++}$ then $A_1 \otimes A_2 \in \mathcal{A}^{++}$.*
2. *If $\omega_1 : \mathcal{A}_1 \to \mathsf{C}$, $\omega_2 : \mathcal{A}_2 \to \mathsf{C}$ are positive linear functionals then $\omega_1 \otimes \omega_2 : \mathcal{A} \to \mathsf{C}$ is a positive linear functional.*

Proof. The first part is trivial. For the second part consider $A_1^{(i)} \in \mathcal{A}_1$, $A_2^{(i)} \in \mathcal{A}_2$ with $i = 1, \ldots, n$. Then

$$\omega_1 \otimes \omega_2 \left(\left(\sum_i A_1^{(i)} \otimes A_2^{(i)} \right)^* \left(\sum_j A_1^{(j)} \otimes A_2^{(j)} \right) \right) = \operatorname{tr}(MN),$$

where the matrices $M, N \in M_n(\mathsf{C})$ are defined by their matrix elements $M_{ij} := \omega_1((A_1^{(i)})^* A_1^{(j)})$ and $N_{ij} := \omega_2((A_2^{(j)})^* A_2^{(i)})$. Then M and N are Hermitian and positive since for $v \in \mathsf{C}^n$ and one clearly has $\langle v, Mv \rangle = \omega_1(A^* A) \geq 0$, where $A = v_1 A_1^{(1)} + \cdots + v_n A_1^{(n)}$ and analogously for N. Then $\operatorname{tr}(MN) \geq 0$ by Corollary A.5. $\square$

Remark 4.4. *Though the tensor product of positive functionals and the tensor product of algebraically positive elements are (algebraically) positive, in the more general case of positive elements $A_1 \in \mathcal{A}_1^+$, $A_2 \in \mathcal{A}_2^+$ there seems to be no simple answer to the question of whether $A_1 \otimes A_2 \in \mathcal{A}^+$. The reason is that in order to establish positivity for $A_1 \otimes A_2$ one has to test $A_1 \otimes A_2$ on all positive linear functionals of $\mathcal{A}$ and not only on the positive linear combinations of factoring ones.*

Consider now *-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ and bimodules ${}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1}$ and ${}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$ out of which we want to construct a ($\mathcal{B}$-$\mathcal{A}$)-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ where $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\mathcal{B} := \mathcal{B}_1 \otimes \mathcal{B}_2$. To this end we set ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} := {}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1} \otimes {}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_1}$ which becomes a ($\mathcal{B}$-$\mathcal{A}$)-bimodule in the usual way. Assume furthermore that ${}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1}$ and ${}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$ are endowed with $\mathcal{A}_1$-valued and $\mathcal{A}_2$-valued inner products, respectively, such that (X1)–(X3) are fulfilled. Then we define an $\mathcal{A}$-valued inner product for ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ by (anti-)linear extension of

$$\langle x \otimes y, x' \otimes y' \rangle_{\mathcal{A}} := \langle x, x' \rangle_{\mathcal{A}_1} \otimes \langle y, y' \rangle_{\mathcal{A}_2}. \tag{4.6}$$

Clearly, (X1)–(X3) are also satisfied for $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ as an easy computation shows. Moreover, if both inner products $\langle \cdot, \cdot \rangle_{\mathcal{A}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}_2}$ satisfy (X5) then $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ satisfies (X5), too. The same is true for the fullness condition (X6).

It remains to check the positivity of $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ under some positivity assumption for $\langle \cdot, \cdot \rangle_{\mathcal{A}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}_2}$. Due to Remark 4.4, one expects this task to be more complicated in general. Nevertheless we can prove the following proposition:

Proposition 4.5. *Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$ be *-algebras over* C *and let* ${}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1}$ *and* ${}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$ *be corresponding bimodules. Then we have for* $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{B} := \mathcal{B}_1 \otimes \mathcal{B}_2$ *and* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = {}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1} \otimes {}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$:

1. ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *is a* $(\mathcal{B}$-$\mathcal{A})$*-bimodule.*
2. If ${}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1}$ *and* ${}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$ *are endowed with* $\mathcal{A}_1$*-valued and* $\mathcal{A}_2$*-valued inner products, respectively, satisfying* (X1)–(X3) *then* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *also carries a canonical* $\mathcal{A}$*-valued inner product which satisfies* (X1)–(X3).
3. *If in addition to* (2) *the inner products* $\langle\cdot,\cdot\rangle_{\mathcal{A}_1}$ *and* $\langle\cdot,\cdot\rangle_{\mathcal{A}_2}$ *satisfy* (X5) *then* $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ *also satisfies* (X5). *The same holds for* (X6).
4. *If in addition to* (2) *the bimodules* ${}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1}$ *and* ${}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$ *satisfy* (X4a) *and* (P1)–(P3) *then* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *also satisfies* (X4a) *and* (P1)–(P3).

Proof. It remains to check the last part. It is straightforward to verify that ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies (P1)–(P3) with the canonically induced direct sum and the corresponding tensor products of the pseudo-cyclic vectors as pseudo-cyclic vectors for the Cartesian product of the corresponding index sets. Using the pseudo-cyclicity as well as (X4a) for each of the given bimodules one finally verifies (X4a) for the new bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ by a lengthy but easy computation. □

Although there may be more general situations, where the tensor product of two such bimodules with inner products yields a bimodule for the corresponding tensor product of the *-algebras, the above construction turns out to be quite useful in Section 6.

Next we shall mention the connection between algebraic Rieffel induction and the GNS construction. Again we follow the well-known situation as in C^*-algebra theory, see, e.g., [51, Chapter IV, Section 2.2].

Let $\omega : \mathcal{A} \to \mathsf{C}$ be a positive linear functional of a *-algebra $\mathcal{A}$ over C. We regard $\mathcal{A} = {}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}}$ as an $\mathcal{A}$-left module and as a C-right module using the left-multiplication by elements of $\mathcal{A}$ on itself and the scalar multiplication by elements in C. Then we consider $\langle\cdot,\cdot\rangle_\omega : {}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}} \times {}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}} \to \mathsf{C}$ defined by $\langle A, B\rangle_\omega := \omega(A^*B)$. It follows immediately that $\langle\cdot,\cdot\rangle_\omega$ is a C-valued inner product for ${}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}}$ which satisfies (X1)–(X5). On the other hand (P1)–(P3) are *not* necessarily fulfilled. Nevertheless in this case we can prove (P) directly.

Lemma 4.6. *Let $\mathcal{A}$ be a *-algebra over* C *and* $\omega : \mathcal{A} \to \mathsf{C}$ *a positive linear functional. Then the* $(\mathcal{A}$-$\mathsf{C})$*-bimodule* ${}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}}$ *endowed with the inner product* $\langle\cdot,\cdot\rangle_\omega$ *induced by* ω *satisfies* (X1)–(X5) *and* (P).

Proof. The verification of (X1)–(X5) is trivial. Thus it remains to show (P). First notice that any *-representation π_{C} of C on a pre-Hilbert space $\mathfrak{H}$ is of the form $\pi_{\mathsf{C}}(z) = zP$, where $P = \pi_{\mathsf{C}}(1)$ is a Hermitian projection, and also that any such projection yields a *-representation of C. Consider $\tilde{\mathfrak{K}} = \mathcal{A} \otimes^\pi \mathfrak{H}$, where the tensor product is now constructed using π. Moreover, let $\psi_1, \ldots, \psi_n \in \mathfrak{H}$ and $A_1, \ldots, A_n \in \mathcal{A}$. Using that $P^2 = P = P^*$, we then have

$$\left\langle \sum_i A_i \otimes \psi_i, \sum_j A_j \otimes \psi_j \right\rangle_{\tilde{\mathfrak{K}}} = \sum_{ij} \omega(A_i^* A_j) \langle P\psi_i, P\psi_j \rangle_{\mathfrak{H}} = \mathrm{tr}(MN),$$

where $M, N \in M_n(\mathsf{C})$ are defined by $M_{ij} := \omega(A_i^* A_j)$ and $N_{ij} := \langle P\psi_j, P\psi_i \rangle_{\mathfrak{H}}$. As in the proof of Lemma 4.3 we notice that M as well as N are Hermitian and positive, whence $\mathrm{tr}(MN) \geq 0$ by Corollary A.5. Thus (P) is shown. $\square$

In order to obtain the GNS representation π_ω of $\mathcal{A}$ as an induced representation we take a particular *-representation of C, namely the *-representation by left-multiplications of C on itself, where the inner product is given by $\langle z, w \rangle = \bar{z}w$. Thus in this case $\tilde{\mathfrak{K}} = \mathcal{A} \otimes \mathsf{C} \cong \mathcal{A}$ and $\langle A, B \rangle_{\tilde{\mathfrak{K}}} = \omega(A^* B)$ canonically. Hence

$$\tilde{\mathfrak{K}}^\perp = \{A \in \mathcal{A} | \omega(B^* A) = 0 \ \ \forall B \in \mathcal{A}\} = \mathcal{J}_\omega$$

coincides with the Gel'fand ideal and thus $\mathfrak{K} = \tilde{\mathfrak{K}} / \tilde{\mathfrak{K}}^\perp = \mathfrak{H}_\omega$ is the correct GNS representation space. Furthermore it is easy to see that in this case the induced representation $\pi_{\mathcal{A}}$ coincides with the GNS representation π_ω. Thus as in C^*-algebra theory the GNS construction is a particular case of Rieffel induction.

Proposition 4.7. *Let $\mathcal{A}$ be a *-algebra over C and $\omega : \mathcal{A} \to \mathsf{C}$ a positive linear functional. Then the GNS representation π_ω coincides with the representation which is Rieffel induced out of the canonical C-representation on itself by means of the $(\mathcal{A}$-$\mathsf{C})$-bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}}$ with inner product given by ω.*

Finally, let us mention the following construction of a bimodule out of a *-homomorphism $\Phi : \mathcal{B} \to \mathcal{A}$. We set ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = {}_{\Phi(\mathcal{B})}\mathcal{A}_{\mathcal{A}}$ with the usual $\mathcal{A}$-right action on itself and the $\mathcal{B}$-left action given by Φ, i.e. $\mathsf{L}_{\mathcal{B}}(B)A := \Phi(B)A$. The $\mathcal{A}$-valued inner product is defined to be

$$\langle A, A' \rangle_{\mathcal{A}} := A^* A', \tag{4.7}$$

and it is easily verified that $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ satisfies (X1)–(X3), (X4a), as well as (X5) since Φ is a *-homomorphism. Moreover, we can verify (P) directly: let $(\mathfrak{H}, \pi_{\mathcal{A}})$ be a *-representation of $\mathcal{A}$ and let $\psi_1, \ldots, \psi_n \in \mathfrak{H}$ and $A_1, \ldots, A_n \in \mathcal{A}$ then

$$\sum_{i,j} \langle A_i \otimes \psi_i, A_j \otimes \psi_j \rangle_{\tilde{\mathfrak{K}}} = \left\langle \sum_i \pi_{\mathcal{A}}(A_i)\psi_i, \sum_j \pi_{\mathcal{A}}(A_j)\psi_j \right\rangle_{\mathfrak{H}} \geq 0$$

clearly shows (P). Moreover, if $\mathcal{A}$ has even an approximate identity $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha \in I}$ then (P1)–(P3) are also fulfilled using the E_α as pseudo-cyclic vectors. Obviously, in this case (X6) is also satisfied.

Proposition 4.8. *Let $\mathcal{A}, \mathcal{B}$ be *-algebras over C and let $\Phi : \mathcal{B} \to \mathcal{A}$ be a *-homomorphism. Then ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = {}_{\Phi(\mathcal{B})}\mathcal{A}_{\mathcal{A}}$* is a *$(\mathcal{B}$-$\mathcal{A})$-bimodule with canonical $\mathcal{A}$-valued inner product satisfying* (X1)–(X3), (X4a), (X5) *and* (P). *If $\mathcal{A}$ has in addition an approximate identity then* (X6) *and* (P1)–(P3) *are also fulfilled.*

If we assume $\mathcal{A}$ to have an approximate identity and π to be a strongly non-degenerate *-representation of $\mathcal{A}$, then the induced representation in Proposition 4.8 is canonically equivalent to the pull-back representation by Φ.

5. Equivalence bimodules and formal Morita equivalence

Given $\mathcal{A}$ and $\mathcal{B}$ *-algebras over C, we saw previously how to construct a *-functor $\mathfrak{R}_{\mathfrak{X}}$: *-$\mathsf{rep}(\mathcal{A}) \to {}^*$-$\mathsf{rep}(\mathcal{B})$ associated to a ($\mathcal{B}$-$\mathcal{A}$)-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ (equipped with some extra structure). In this section, we will be concerned with the question of how to define bimodules that give rise to equivalence of categories.

First note that to each given ($\mathcal{B}$-$\mathcal{A}$)-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, there naturally corresponds an ($\mathcal{A}$-$\mathcal{B}$)-bimodule ${}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}}$, defined as in the theory of C^*-algebras (see [64,65]). We let $\overline{\mathfrak{X}}$ be the C-module conjugate to $\mathfrak{X}$: as an additive group, we have $\overline{\mathfrak{X}} = \mathfrak{X}$, but if $^- : \mathfrak{X} \to \overline{\mathfrak{X}}, x \mapsto \overline{x}$ denotes the identity map, we define the scalar multiplication on $\overline{\mathfrak{X}}$ by $a\overline{x} = \overline{\overline{a}x},\ a \in \mathsf{C}$. We then define a left $\mathcal{A}$-action and a right $\mathcal{B}$-action on $\overline{\mathfrak{X}}$ by

$$A\bar{x} = \overline{xA^*}, \qquad \bar{x}B = \overline{B^*x} \ \text{ for } \ A \in \mathcal{A},\ B \in \mathcal{B}.$$

If $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ is a positive semi-definite $\mathcal{B}$-valued inner product on ${}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}}$ satisfying (X1)–(X5) and (P), then we can consider the corresponding functor $\mathfrak{R}_{\overline{\mathfrak{X}}}$: *-$\mathsf{rep}(\mathcal{B}) \to {}^*$-$\mathsf{rep}(\mathcal{A})$, which is a natural candidate for the inverse of $\mathfrak{R}_{\mathfrak{X}}$. Observe that the existence of such a $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ is equivalent to the existence of a positive semi-definite $\mathcal{B}$-valued inner product on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, defined by ${}_{\mathcal{B}}\langle x, y\rangle = \langle\overline{x},\overline{y}\rangle,\ x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfying:

(Y1) ${}_{\mathcal{B}}\langle ax + by, z\rangle = a\,{}_{\mathcal{B}}\langle x, z\rangle + b\,{}_{\mathcal{B}}\langle y, z\rangle$,
(Y2) ${}_{\mathcal{B}}\langle x, y\rangle = {}_{\mathcal{B}}\langle y, x\rangle^*$,
(Y3) ${}_{\mathcal{B}}\langle B \cdot x, y\rangle = B\,{}_{\mathcal{B}}\langle x, y\rangle$,
(Y4) ${}_{\mathcal{B}}\langle x, x\rangle \geq 0$,
(Y5) ${}_{\mathcal{B}}\langle x \cdot A, y\rangle = {}_{\mathcal{B}}\langle x, y \cdot A^*\rangle$

for all $x, y, z \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, $a, b \in \mathsf{C}$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$. It is also clear that $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ satisfies (X6) if and only if ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ satisfies:

(Y6) $\mathcal{B} = \mathsf{C}\text{-span}\{{}_{\mathcal{B}}\langle x, y\rangle | x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}\}$,

also observe that $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ satisfies (X4a), (X4′) and (X4a′) if and only if ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ satisfies the corresponding conditions:

(Y4a) ${}_{\mathcal{B}}\langle x, x\rangle \in \mathcal{B}^{++}$,
(Y4′) ${}_{\mathcal{B}}\langle x, x\rangle \geq 0$ and ${}_{\mathcal{B}}\langle x, x\rangle = 0$ implies $x = 0$,
(Y4a′) ${}_{\mathcal{B}}\langle x, x\rangle \in \mathcal{B}^{++}$ and ${}_{\mathcal{B}}\langle x, x\rangle = 0$ implies $x = 0$,

for all $x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. Moreover, we define the following property:

(Q) We say that ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies property (Q) if $({}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}}, \langle\cdot,\cdot\rangle_{\mathcal{B}})$ satisfies property (P).

We are now ready for the following definition.

Definition 5.1. A ($\mathcal{B}$-$\mathcal{A}$)-equivalence bimodule is a ($\mathcal{B}$-$\mathcal{A}$)-bimodule in the sense of (3.1) with the following additional structure:

(E1) An $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ satisfying (X1)–(X6).
(E2) A $\mathcal{B}$-valued inner product ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ satisfying (Y1)–(Y6).
(E3) The compatibility condition ${}_{\mathcal{B}}\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_{\mathcal{A}}$, $x, y, z \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$.
(E4) ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies both properties (P) and (Q).

We give a set of sufficient conditions to guarantee that property (Q) holds analogous to conditions (P1)–(P3) for the $\mathcal{B}$-action on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$.

(Q1) ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = \oplus_{j\in J}\mathfrak{X}^{(j)}$ and $\mathfrak{X}^{(k)} \perp \mathfrak{X}^{(j)}$ for all $k \neq j \in J$ with respect to ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$.
(Q2) The left $\mathcal{B}$-action $\mathsf{L}_{\mathcal{B}}$ preserves this direct sum.
(Q3) Each $\mathfrak{X}^{(j)}$ is pseudo-cyclic for $\mathsf{L}_{\mathcal{B}}$ with filtered subspaces $\mathfrak{X}^{(j)} = \bigcup_{\beta\in J^{(j)}}\mathfrak{X}^{(j)}_{\beta}$ and pseudo-cyclic vectors $\Omega^{(j)}_{\beta}$.

Remark 5.2. *The conditions* (Q1)–(Q3) *are independent of* (P1)–(P3) *and we do not require any compatibility between the right $\mathcal{A}$-action $\mathsf{R}_{\mathcal{A}}$ and* (Q1)–(Q3) *nor between the left $\mathcal{B}$-action $\mathsf{L}_{\mathcal{B}}$ and* (P1)–(P3).

It is then clear that a bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfying (E1)–(E3) and (P1)–(P3), (Q1)–(Q3) is an equivalence bimodule.

Definition 5.3. $\mathcal{A}$ and $\mathcal{B}$ are called (formally) Morita equivalent if there exists a ($\mathcal{B}$-$\mathcal{A}$)-equivalence bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$.

Whenever the context is clear, we will refer to formal Morita equivalence simply as Morita equivalence. From the definitions, we see that if ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is an equivalence bimodule, so is ${}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}}$; hence Morita equivalence is a symmetric relation. We will next discuss reflexivity and transitivity.

Proposition 5.4. *Suppose $\mathcal{A}$ is a *-algebra over C with an approximate identity $\{E_\alpha, \mathcal{A}_\alpha\}_{\alpha\in I}$. Let $\mathcal{B}$ be a *-algebra over C and suppose $\Phi : \mathcal{B} \to \mathcal{A}$ is an isomorphism. Then $\mathcal{A}$ and $\mathcal{B}$ are* (*formally*) *Morita equivalent. In particular, $\mathcal{A}$ is Morita equivalent to itself.*

Proof. Consider the ($\mathcal{B}$-$\mathcal{A}$)-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = {}_{\Phi(\mathcal{B})}\mathcal{A}_{\mathcal{A}}$ as defined in Proposition 4.8 and define on this bimodule a $\mathcal{B}$-valued inner product given by ${}_{\mathcal{B}}\langle A_1, A_2\rangle = \Phi^{-1}(A_1A_2^*)$. Then, just as in Proposition 4.8, one can show that the axioms (X1)–(X3), (X4a), (X5), (X6) and (P1)–(P3) hold, as well as (Y1)–(Y3), (Y4a), (Y5), (Y6) and (Q1)–(Q3). Finally, a simple computation shows that (E3) also holds. □

We will now discuss transitivity properties of Morita equivalence. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be *-algebras over C. Suppose $\mathcal{B}$ and $\mathcal{A}$ are Morita equivalent, with equivalence bimodule

${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, and also that $\mathcal{A}$ and $\mathcal{C}$ are Morita equivalent, with equivalence bimodule ${}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$. Before we state the main result, we need the following observation:

Lemma 5.5. *Let $A \in \mathcal{A}$ be positive. Then for all $x' \in {}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$ we have $\langle x', Ax'\rangle_{\mathcal{C}} \in \mathcal{C}^+$.*

Proof. Let $\omega : \mathcal{C} \to \mathsf{C}$ be a positive functional. Fix $x' \in {}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$ and consider the linear functional $\hat{\omega} : \mathcal{A} \to \mathsf{C}$ on $\mathcal{A}$, defined by $\hat{\omega}(A) = \omega(\langle x', Ax'\rangle_{\mathcal{C}})$. It is clear that $\hat{\omega}(A^*A) \geq 0$ (by (X4)) for all $A \in \mathcal{A}$ and hence $\hat{\omega}$ is positive. So if A is positive, $\omega(\langle x', Ax'\rangle_{\mathcal{C}}) \geq 0$ for all ω positive and the proof is complete. □

Proposition 5.6. *Suppose that ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies* (P1)–(P3) *and ${}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$ satisfies* (Q1)–(Q3). *Then $\mathcal{B}$ and $\mathcal{C}$ are also Morita equivalent.*

Proof. Let $\mathfrak{X}'' = \mathfrak{X} \otimes_{\mathcal{A}} \mathfrak{X}'$ be the ($\mathcal{A}$ balanced) tensor product of ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$. It has a natural ($\mathcal{B}$-$\mathcal{C}$)-bimodule structure, and we denote it by ${}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}}$. Note that the formula

$$\langle\langle x_1 \otimes x_1', x_2 \otimes x_2'\rangle\rangle_{\mathcal{C}} = \langle x_1', \langle x_1, x_2\rangle_{\mathcal{A}} \cdot x_2'\rangle_{\mathcal{C}}$$

uniquely defines a map $\langle\langle \cdot, \cdot\rangle\rangle_{\mathcal{C}} : {}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}} \times {}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}} \to \mathsf{C}$ satisfying (X1)–(X3) and (X5). Similarly,

$${}_{\mathcal{B}}\langle\langle x_1 \otimes x_1', x_2 \otimes x_2'\rangle\rangle = {}_{\mathcal{B}}\langle x_1 \cdot_{\mathcal{A}} \langle x_1', x_2'\rangle, x_2\rangle$$

uniquely defines a map ${}_{\mathcal{B}}\langle\langle \cdot, \cdot\rangle\rangle : {}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}} \times {}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}} \to \mathsf{C}$ satisfying (Y1)–(Y3) and (Y5). Let us show that $\langle\langle \cdot, \cdot\rangle\rangle_{\mathcal{C}}$ satisfies (X4). Recall that since ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies (P1)–(P3), any $z \in {}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}}$ can be written as

$$z = \sum_i x_1^{(i)} \otimes x_1' + \cdots + x_n^{(i)} \otimes x_n', \quad x_1^{(i)}, \ldots, x_n^{(i)} \in \mathfrak{X}^{(i)}.$$

But following conditions (P1)–(P3), we know that for each i, there exists an α_i such that $x_1^{(i)}, \ldots, x_n^{(i)} \in \mathfrak{X}_{\alpha_i}^{(i)}$. So, we can find $A_1^{(i)}, \ldots, A_n^{(i)} \in \mathcal{A}$ such that $x_1^{(i)} = \Omega_{\alpha_i}^{(i)} \cdot A_1^{(i)}, \ldots, x_n^{(i)} = \Omega_{\alpha_i}^{(i)} \cdot A_n^{(i)}$ and hence

$$z = \sum_i \Omega_{\alpha_i}^{(i)} \cdot A_1^{(i)} \otimes x_1' + \cdots + \Omega_{\alpha_i}^{(i)} \cdot A_n^{(i)} \otimes x_n' = \sum_i \Omega_{\alpha_i}^{(i)} \otimes y_i,$$

where $y_i = A_1^{(i)} \cdot x_1' + \cdots + A_n^{(i)} \cdot x_n'$. Therefore, we have

$$\langle\langle z, z\rangle\rangle_{\mathcal{C}} = \sum_{i,j} \left\langle\left\langle \Omega_{\alpha_i}^{(i)} \otimes y_i, \Omega_{\alpha_j}^{(j)} \otimes y_j \right\rangle\right\rangle_{\mathcal{C}} = \sum_{i,j} \left\langle y_i, \left\langle \Omega_{\alpha_i}^{(i)}, \Omega_{\alpha_j}^{(j)} \right\rangle \cdot y_j \right\rangle_{\mathcal{C}}.$$

But since $\mathfrak{X}^{(i)} \perp \mathfrak{X}^{(j)}$ for all $i \neq j$ with respect to $\langle \cdot, \cdot\rangle_{\mathcal{A}}$, it follows that $\langle\langle z, z\rangle\rangle_{\mathcal{C}} = \sum_i \langle y_i, \langle \Omega_{\alpha_i}^{(i)}, \Omega_{\alpha_i}^{(i)}\rangle_{\mathcal{A}} y_i\rangle_{\mathcal{C}}$ and hence $\langle\langle z, z\rangle\rangle_{\mathcal{C}} \geq 0$ by Lemma 5.5. Similarly, we can use that ${}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$ satisfies (Q1)–(Q3) to show that ${}_{\mathcal{B}}\langle\langle \cdot, \cdot\rangle\rangle$ is positive semi-definite. So we conclude that $\langle\langle \cdot, \cdot\rangle\rangle_{\mathcal{C}}$ satisfies (X1)–(X5) and ${}_{\mathcal{B}}\langle\langle \cdot, \cdot\rangle\rangle$ satisfies (Y1)–(Y5). We also observe that it follows from Remark 3.7 that the actions of $\mathcal{A}$ on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$ are strongly non-degenerate and an easy computation, like in the case of C^*-algebras, shows that this implies the fullness conditions (X6) and (Y6). A straightforward computation, also similar to the C^*-algebra setting, shows that the compatibility condition (E3) is also satisfied.

So it only remains to check (E4) to conclude the proof. Let $(\mathfrak{K}, \pi_{\mathcal{C}})$ be a *-representation of $\mathcal{C}$. Then, since ${}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$ satisfies (P), we can define a positive semi-definite Hermitian product on $\tilde{\mathfrak{H}} = {}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}} \otimes_{\mathcal{C}} \mathfrak{K}$ by

$$\langle x_1' \otimes k_1, x_2' \otimes k_2\rangle_{\tilde{\mathfrak{H}}} = \langle k_1, \pi_{\mathcal{C}}(\langle x_1', x_2'\rangle_{\mathcal{C}})k_2\rangle_{\mathfrak{K}}, \quad x_1', x_2' \in {}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}, \ \ k_1, k_2 \in \mathfrak{K}.$$

But now $\tilde{\mathfrak{H}}$ is a C-module with a positive semi-definite Hermitian product and $\mathcal{A}$ acts on it by adjointable operators. So due to Lemma 3.2, we can define a positive semi-definite Hermitian product on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \tilde{\mathfrak{H}} =_{\mathcal{B}} \mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}} \otimes_{\mathcal{C}} \mathfrak{K})$ by setting

$$\begin{aligned}\langle x_1 \otimes (x_1' \otimes k_1), x_2 \otimes (x_2' \otimes k_2)\rangle &= \langle x_1 \otimes k_1, (\langle x_1, x_2\rangle_{\mathcal{A}} \cdot x_2') \otimes k_2\rangle_{\tilde{\mathfrak{H}}} \\ &= \langle k_1, \pi_{\mathcal{C}}(\langle x_1', \langle x_1, x_2\rangle_{\mathcal{A}} x_2'\rangle_{\mathcal{C}})k_2\rangle_{\mathfrak{K}}.\end{aligned}$$

Finally, note that the last expression is just the definition of the Hermitian product induced on $({}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}) \otimes_{\mathcal{C}} \mathfrak{K} = {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}} \otimes_{\mathcal{C}} \mathfrak{K})$ by the bimodule ${}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}} = {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathfrak{X}'_{\mathcal{C}}$. So ${}_{\mathcal{B}}\mathfrak{X}''_{\mathcal{C}}$ satisfies property (P). Analogously, ${}_{\mathcal{C}}\overline{\mathfrak{X}}''_{\mathcal{B}}$ also satisfies (P), for we can identify ${}_{\mathcal{C}}\overline{\mathfrak{X}}''_{\mathcal{B}} \cong {}_{\mathcal{C}}\overline{\mathfrak{X}}'_{\mathcal{A}} \otimes {}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}}$. □

It is important to point out that Proposition 5.6 does not show transitivity in general, but it will still be useful later, in Section 6. We will finish this section with a discussion about functors corresponding to equivalence bimodules. We will start with two lemmas which are analogous to results in C^*-algebras.

Lemma 5.7. *Suppose $\mathcal{A}, \mathcal{B}$ are *-algebras over C and let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be an equivalence bimodule. Let $(\mathfrak{H}, \pi_{\mathcal{A}})$ be a strongly non-degenerate *-representation of $\mathcal{A}$. Then $\mathfrak{R}_{\overline{\mathfrak{X}}} \circ \mathfrak{R}_{\mathfrak{X}}(\mathfrak{H}, \pi_{\mathcal{A}})$ is unitarily equivalent to $(\mathfrak{H}, \pi_{\mathcal{A}})$. Analogously, if $(\mathfrak{K}, \pi_{\mathcal{B}})$ is a strongly non-degenerate *-representation of $\mathcal{B}$, then $\mathfrak{R}_{\mathfrak{X}} \circ \mathfrak{R}_{\overline{\mathfrak{X}}}(\mathfrak{K}, \pi_{\mathcal{B}})$ is unitarily equivalent to $(\mathfrak{K}, \pi_{\mathcal{B}})$.*

Proof. The proof basically follows [64, Section 3.3]. Let $\tilde{\mathfrak{K}} = {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}$ and $\mathfrak{K} = \tilde{\mathfrak{K}}/(\tilde{\mathfrak{K}})^{\perp}$. Also define $\tilde{\mathfrak{H}}' = {}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathfrak{K}$ and $\mathfrak{H}' = \tilde{\mathfrak{H}}'/(\mathfrak{H}')^{\perp}$. Note that there is a linear map $U : \mathfrak{H} \to \mathfrak{H}'$ uniquely defined by

$$U([\bar{x} \otimes [y \otimes \psi]]) = \pi_{\mathcal{A}}(\langle x, y\rangle_{\mathcal{A}})\psi \ \text{ for } \ x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}, \ \ \psi \in \mathfrak{H}.$$

Since $\pi_{\mathcal{A}}$ is strongly non-degenerate and $\langle \cdot, \cdot\rangle_{\mathcal{A}}$ is full, it immediately follows that U is onto. A simple computation using the definitions shows that U preserves the Hermitian products, and therefore it is unitary. It is also easy to check that U intertwines $\pi_{\mathcal{A}}$ and $\mathfrak{R}_{\overline{\mathfrak{X}}} \circ \mathfrak{R}_{\mathfrak{X}}(\pi_{\mathcal{A}})$. Thus the conclusion follows. The same argument holds for $\mathcal{B}$. □

Moreover, the previous construction is natural in the following sense.

Lemma 5.8. *Suppose we have two strongly non-degenerate* **-representations* $(\mathfrak{H}_1, \pi^1_{\mathcal{A}})$ *and* $(\mathfrak{H}_2, \pi^2_{\mathcal{A}})$ *of* $\mathcal{A}$, *and let* $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ *be an intertwiner operator* (*adjointable or isometric*). *Let* $U_1 : \mathfrak{R}_{\overline{\mathfrak{X}}} \circ \mathfrak{R}_{\mathfrak{X}}(\mathfrak{H}_1) \to \mathfrak{H}_1$ and $U_2 : \mathfrak{R}_{\overline{\mathfrak{X}}} \circ \mathfrak{R}_{\mathfrak{X}}(\mathfrak{H}_2) \to \mathfrak{H}_2$ *be the two unitary equivalences as in Lemma* 5.7. *Then* $U_2 \circ (\mathfrak{R}_{\overline{\mathfrak{X}}} \circ \mathfrak{R}_{\mathfrak{X}}(T)) = T \circ U_1$. *An analogous statement holds for* $\mathcal{B}$.

Proof. This is also a simple computation using the definitions, that can be carried out just like in the C^*-algebra setting (see [64, Section 3.3]). □

Before we state the main theorem about equivalence of categories, we need the following definition.

Definition 5.9. We call an equivalence bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ non-degenerate if the actions $\mathsf{L}_{\mathcal{B}}$ and $\mathsf{R}_{\mathcal{A}}$ are both strongly non-degenerate.

It follows from Proposition 3.6 that if ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is a non-degenerate ($\mathcal{B}$-$\mathcal{A}$)-equivalence bimodule then it makes sense to restrict the induced functors $\mathfrak{R}_{\mathfrak{X}}, \mathfrak{R}_{\overline{\mathfrak{X}}}$ to strongly non-degenerate representations:

$$\mathfrak{R}_{\mathfrak{X}} : {}^*\text{-}\mathsf{Rep}(\mathcal{A}) \to {}^*\text{-}\mathsf{Rep}(\mathcal{B}) \qquad \mathfrak{R}_{\overline{\mathfrak{X}}} : {}^*\text{-}\mathsf{Rep}(\mathcal{B}) \to {}^*\text{-}\mathsf{Rep}(\mathcal{A}). \tag{5.1}$$

We can then state the following theorem.

Theorem 5.10. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* **-algebras over* C. *If* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *is a non-degenerate* ($\mathcal{B}$-$\mathcal{A}$)-*equivalence bimodule then* $\mathfrak{R}_{\mathfrak{X}}$ *and* $\mathfrak{R}_{\overline{\mathfrak{X}}}$ *define an equivalence of categories between* *-$\mathsf{Rep}(\mathcal{A})$ *and* *-$\mathsf{Rep}(\mathcal{B})$.

The proof is a direct consequence of Lemmas 5.7 and 5.8. Let us discuss some situations where an equivalence bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is automatically non-degenerate. Observe that this is clearly the case if $\mathcal{A}$ and $\mathcal{B}$ are unital and $\mathsf{L}_{\mathcal{B}}(1_{\mathcal{B}}) = \mathsf{R}_{\mathcal{A}}(1_{\mathcal{A}}) = \mathrm{id}$ (see Remark 5.18). We will now need the following lemma.

Lemma 5.11. *Let* $\mathcal{B}$ *be a* **-algebra over* C *with approximate identity and let* ${}_{\mathcal{B}}\mathfrak{X}$ *be a left* $\mathcal{B}$*-module equipped with a* $\mathcal{B}$*-valued positive definite inner product. Then the action of* $\mathcal{B}$ *on* ${}_{\mathcal{B}}\mathfrak{X}$ *is strongly non-degenerate. The same holds for right* $\mathcal{A}$*-modules with a corresponding* $\mathcal{A}$*-valued positive definite inner product* $\langle \cdot, \cdot \rangle_{\mathcal{A}}$.

Proof. Note that for a general $B \in \mathcal{B}$, we have

$${}_{\mathcal{B}}\langle x - B \cdot x, x - B \cdot x \rangle = {}_{\mathcal{B}}\langle x, x \rangle - B\, {}_{\mathcal{B}}\langle x, x \rangle - {}_{\mathcal{B}}\langle x, x \rangle B^* + B\, {}_{\mathcal{B}}\langle x, x \rangle B^*.$$

But since $\mathcal{B}$ has an approximate identity, we can find $E_\alpha \in \mathcal{B}$ such that $E_\alpha {}_{\mathcal{B}}\langle x, x \rangle = {}_{\mathcal{B}}\langle x, x \rangle E_\alpha = {}_{\mathcal{B}}\langle x, x \rangle$ and $E_\alpha = E_\alpha^*$. So, for $B = E_\alpha$ we get ${}_{\mathcal{B}}\langle x - E_\alpha \cdot x, x - E_\alpha \cdot x \rangle = 0$ and by non-degeneracy of ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ it follows that $x = E_\alpha \cdot x$. The same argument can be applied to right $\mathcal{A}$-modules. □

We then have the following result.

Corollary 5.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be* **-algebras over* C *with approximate identities and suppose* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *is a* ($\mathcal{B}$-$\mathcal{A}$)*-equivalence bimodule satisfying* (X4$'$) *and* (Y4$'$). *Then* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *is non-degenerate. In particular, the induced functors* $\mathfrak{R}_{\mathfrak{X}}$ *and* $\mathfrak{R}_{\overline{\mathfrak{X}}}$ *define an equivalence of categories between* *-$\mathsf{Rep}(\mathcal{A})$ *and* *-$\mathsf{Rep}(\mathcal{B})$.

From Remark 3.7, we note that if ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is an equivalence bimodule satisfying (P1)–(P3), (Q1)–(Q3), then the actions $\mathsf{L}_{\mathcal{B}}$ and $\mathsf{R}_{\mathcal{A}}$ are strongly non-degenerate. The following corollay follows immediately.

Corollary 5.13. *If* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *is an equivalence bimodule satisfying* (P1)–(P3), (Q1)–(Q3), *then* $\mathfrak{R}_{\mathfrak{X}}$ *and* $\mathfrak{R}_{\overline{\mathfrak{X}}}$ *define an equivalence of categories between* *-$\mathsf{Rep}(\mathcal{A})$ *and* *-$\mathsf{Rep}(\mathcal{B})$.

We call two *-algebras over C “categorically” Morita equivalent if they have equivalent categories of strongly non-degenerate representations. Note that Theorem 5.10 shows that formal Morita equivalence (through a non-degenerate equivalence bimodule) implies “categorical” Morita equivalence. A natural question is then whether or not these two notions are equivalent. We will now see that, as in the theory of C^*-algebras (see [10,67]), this is not the case. To this end, we will consider C and $\bigwedge(\mathsf{C}^n)$, the Grassmann algebra of C^n. We define a *-involution on $\bigwedge(\mathsf{C}^n)$ by setting $1^* = 1$ and $e_i^* = e_i$ for all $i = 1, \dots, n$, where $e_1, \dots, e_n$ is the canonical basis of C^n (see, e.g., [23, Section 2] for a discussion about this *-algebra). Let now $(\mathfrak{H}, \pi)$ be a strongly non-degenerate *-representation of $\bigwedge(\mathsf{C}^n)$. Since $\pi(e_i)$ is self-adjoint and nilpotent (for $e_1 \wedge e_1 = 0$), it follows from Proposition 2.8 that $\pi(e_i) = 0$ for all $i = 1, \dots, n$ and hence $\pi(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \pi(e_{i_1}) \dots \pi(e_{i_r}) = 0$ for all $r \geq 1$ and $i_j \in \mathbb{N}$ (and $\pi(1) = \mathrm{id}$ by non-degeneracy). If we think of C as embedded in $\bigwedge(\mathsf{C}^n)$ in the natural way, we can then conclude that any strongly non-degenerate *-representation of C extends uniquely to a strongly non-degenerate *-representation of $\bigwedge(\mathsf{C}^n)$ and it is also clear that any such representation of $\bigwedge(\mathsf{C}^n)$ can be restricted to C. It is easy to check that this correspondence actually establishes an equivalence of categories between *-$\mathsf{Rep}(\mathsf{C})$ and *-$\mathsf{Rep}(\bigwedge(\mathsf{C}^n))$. Hence we have the following proposition.

Proposition 5.14. *-$\mathsf{Rep}(\mathsf{C})$ *and* *-$\mathsf{Rep}(\bigwedge(\mathsf{C}^n))$ *are equivalent categories*.

We will now show, however, that C and $\bigwedge(\mathsf{C}^n)$ are not formally Morita equivalent. In order to do that, we need to observe a couple of general results about equivalence bimodules. So let $\mathcal{A}$ and $\mathcal{B}$ be two *-algebras over C.

Lemma 5.15. *Let* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *be a* ($\mathcal{B}$-$\mathcal{A}$)*-bimodule satisfying* (E2). *If* $\mathcal{B}$ *has an approximate identity, then the action map* $\mathsf{L}_{\mathcal{B}} : \mathcal{B} \to \mathrm{End}_{\mathcal{A}}({}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}})$ *is injective. If* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *satisfies* (E1), *then an analogous statement holds for* $\mathcal{A}$ *and* $\mathsf{R}_{\mathcal{A}}$.

Proof. Suppose $\mathsf{L}_{\mathcal{B}}(B) = 0$. Since $\mathcal{B}$ has an approximate identity, there exists $E_\alpha \in \mathcal{B}$ such that $B = E_\alpha B$ and using that ${}_{\mathcal{B}}\langle\cdot, \cdot\rangle$ is full, we can write $E_\alpha = {}_{\mathcal{B}}\langle x_1, y_1\rangle + \cdots + {}_{\mathcal{B}}\langle x_n, y_n\rangle$.

So

$$B=B({}_{\mathcal{B}}\langle x_1, y_1\rangle + \cdots + {}_{\mathcal{B}}\langle x_n, y_n\rangle)= {}_{\mathcal{B}}\langle \mathsf{L}_{\mathcal{B}}(B)x_1, y_1\rangle+\cdots+ {}_{\mathcal{B}}\langle \mathsf{L}_{\mathcal{B}}(B)x_n, y_n\rangle=0,$$

since we are assuming that $\mathsf{L}_{\mathcal{B}}(B) = 0$. The same argument applies to $\mathcal{A}$ and $\mathsf{R}_{\mathcal{A}}$. □

We now define

$$N_{\mathcal{A}} = \{x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} | \langle x, y\rangle_{\mathcal{A}} = 0\, \forall y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}\} \text{ and}$$
$$N_{\mathcal{B}} = \{x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} | {}_{\mathcal{B}}\langle x, y\rangle = 0\, \forall y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}\}, \tag{5.2}$$

and observe the following proposition.

Proposition 5.16. *Let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be a* ($\mathcal{B}$-$\mathcal{A}$)*-bimodule satisfying* (E1)–(E3) *and assume $\mathcal{A}$ and $\mathcal{B}$ have approximate identities. Then $N_{\mathcal{A}} = N_{\mathcal{B}} = N$ and ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}/N$ is still a* ($\mathcal{B}$-$\mathcal{A}$)*-bimodule satisfying* (E1)–(E3). *Furthermore, if ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is an equivalence bimodule, then so is ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}/N$.*

Proof. Suppose $x \in N_{\mathcal{A}}$ and let $y, z \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. Then note that $\mathsf{L}_{\mathcal{B}}({}_{\mathcal{B}}\langle y, x\rangle)z = y\mathsf{R}_{\mathcal{A}}(\langle x, z\rangle_{\mathcal{A}}) = 0$. Since z is arbitrary, it follows that $\mathsf{L}_{\mathcal{B}}({}_{\mathcal{B}}\langle y, x\rangle) = 0$ and hence Lemma 5.15 implies that ${}_{\mathcal{B}}\langle y, x\rangle = 0$ for all $y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. So $x \in N_{\mathcal{B}}$. We can then reverse the argument and conclude that $N_{\mathcal{A}} = N_{\mathcal{B}}$. It is not hard to check that ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}/N$ still carries a natural left $\mathcal{B}$-action and a right $\mathcal{A}$-action (since N is both $\mathcal{A}$ and $\mathcal{B}$ invariant). Moreover, we can also define $\mathcal{A}$- and $\mathcal{B}$-valued inner products on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}/N$ in the natural way and a simple computation shows that all the properties of an equivalence bimodule still hold. □

Remark 5.17. *Observe that it is not necessarily true that $N_{\mathcal{A}} = \{x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} | \langle x, x\rangle_{\mathcal{A}} = 0\}$. Thus, the induced $\mathcal{A}$-valued inner product on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}/N$ does not necessarily satisfy* (X4′) (*and similarly, the induced $\mathcal{B}$-valued inner product on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}/N$ does not necessarily satisfy* (Y4′)). *However, there are important situations where the induced inner product on the quotient is in fact strictly positive, see Lemma* 5.21.

Remark 5.18. *Suppose $\mathcal{A}$ is unital. In this case, we observe that if ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is such that $\langle x, y\rangle_{\mathcal{A}} = 0$ for all $y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ implies that $x = 0$, then $\mathsf{R}_{\mathcal{A}}(1_{\mathcal{A}}) = \mathrm{id}$. To see that, just note that $\langle x, y\rangle_{\mathcal{A}} = \langle x, y\rangle_{\mathcal{A}} \cdot 1_{\mathcal{A}} = \langle x, y \cdot 1_{\mathcal{A}}\rangle_{\mathcal{A}}$* for all *$x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and clearly the analogous statement for $\mathcal{B}$ and $\mathsf{L}_{\mathcal{B}}$ also holds. Hence, it follows from Proposition* 5.16 *that we can assume, without loss of generality, that $\mathsf{R}_{\mathcal{A}}(1_{\mathcal{A}}) = \mathsf{L}_{\mathcal{B}}(1_{\mathcal{B}}) = \mathrm{id}$.*

We can now prove the following proposition.

Proposition 5.19. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are *-algebras with approximate identities, and assume $\mathcal{A}$ has sufficiently many positive linear functionals. Let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be a* ($\mathcal{B}$-$\mathcal{A}$)*-bimodule satisfying* (E1)–(E3). *Then $\mathcal{B}$ also has sufficiently many positive linear functionals.*

Proof. By Proposition 5.16, we can assume that $\langle\cdot, \cdot\rangle_{\mathcal{A}}$ satisfies $\langle x, y\rangle_{\mathcal{A}} = 0$ for all $y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ implies that $x = 0$. Let ω be a positive linear functional in $\mathcal{A}$ and $x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. Note that the map $B \mapsto \omega(\langle x, B \cdot x\rangle_{\mathcal{A}})$ defines a positive linear functional in $\mathcal{B}$. Let $B = B^* \in \mathcal{B}$.

To show that $\mathcal{B}$ has sufficiently many positive linear functionals, it suffices to show that if $B \neq 0$, then there exists ω and x such that $\omega(\langle x, B \cdot x\rangle_{\mathcal{A}}) \neq 0$. To see that, suppose that for all $x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and ω positive linear functional in $\mathcal{A}$, we have $\omega(\langle x, B \cdot x\rangle_{\mathcal{A}}) = 0$. Then since $\mathcal{A}$ has sufficiently many positive linear functionals, it follows that $\langle x, B \cdot x\rangle_{\mathcal{A}} = 0$ for all x. But then, by polarization, it follows that $4\langle x, B \cdot y\rangle_{\mathcal{A}} = 0$ for all $x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, and hence $\langle x, B \cdot y\rangle_{\mathcal{A}} = 0$ for all $x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ since $\mathcal{A}$ is torsion-free (see Proposition 2.8). But then we must have $B \cdot x = 0$ for all $x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and hence by Lemma 5.15 we conclude that $B = 0$. This finishes the proof. □

It is easy to check that $\bigwedge(\mathsf{C}^n)$ is *not* an algebra with sufficiently many positive linear functionals (see, e.g., [23, Section 2]) We then have the following immediate corollary.

Corollary 5.20. *The* **-algebras* C *and* $\bigwedge(\mathsf{C}^n)$ *are not formally Morita equivalent.*

Finally, we will show that if two *-algebras with sufficiently many positive linear functionals (and approximate identities) are formally Morita equivalent, then there actually exists an equivalence bimodule satisfying (X4′) and (Y4′). This will be an immediate consequence of the following lemma.

Lemma 5.21. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* **-algebras with sufficiently many positive linear functionals and approximate identities. Let* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *be an equivalence bimodule. Then*

$$N_{\mathcal{A}} = \{x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} | \langle x, x\rangle_{\mathcal{A}} = 0\}.$$

In particular, there is a well-defined strictly positive $\mathcal{A}$*-valued inner product on* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}/N_{\mathcal{A}}$. *An analogous statement holds for* $N_{\mathcal{B}}$ *and* ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$.

Proof. Note that given a positive linear functional ω, we can define a positive semi-definite Hermitian product on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ by $(x, y) \mapsto \omega(\langle x, y\rangle_{\mathcal{A}})$. It then follows from (2.3) that

$$\omega(\langle x, y\rangle_{\mathcal{A}})\overline{\omega(\langle x, y\rangle_{\mathcal{A}})} \leq \omega(\langle x, x\rangle_{\mathcal{A}})\omega(\langle y, y\rangle_{\mathcal{A}}). \tag{5.3}$$

So, if $\langle x, x\rangle_{\mathcal{A}} = 0$ it follows that $\omega(\langle x, y\rangle_{\mathcal{A}}) = 0$ for all positive linear functional ω. Hence, by Corollary 2.9, we have that $\langle x, y\rangle_{\mathcal{A}} = 0$ for all $y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. The conclusion is now immediate and the same argument can be used for ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$. □

Then we can state the following result, which follows from Lemma 5.21 and Proposition 5.16.

Proposition 5.22. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* **-algebras with sufficiently many positive linear functionals and approximate identities and suppose they are formally Morita equivalent. Then there exists a* ($\mathcal{B}$-$\mathcal{A}$)*-equivalence bimodule satisfying* (X4′) *and* (Y4′).

Note that it follows immediately from Corollary 5.12 that if $\mathcal{A}$ and $\mathcal{B}$ are *-algebras with sufficiently many positive linear functionals and approximate identities which are Morita equivalent, then *-$\mathsf{Rep}(\mathcal{A})$ and *-$\mathsf{Rep}(\mathcal{B})$ are equivalent categories.

6. Formal Morita equivalence for matrix algebras and full projections

We will start this section by discussing how (formally) Morita equivalent *-algebras can be constructed out of each other, in analogy with the theory of C^*-algebras (see [64,65]).

Let $\mathcal{A}$ be a *-algebra over C and let $\mathfrak{X}_{\mathcal{A}}$ be a right $\mathcal{A}$-module equipped with a positive semi-definite $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$. Then we can consider the set of all endomorphisms of $\mathfrak{X}_{\mathcal{A}}$ (that is, right $\mathcal{A}$-linear maps), denoted $\mathrm{End}_{\mathcal{A}}(\mathfrak{X}_{\mathcal{A}})$ and define

$$\mathfrak{B}(\mathfrak{X}_{\mathcal{A}}) = \{T \in \mathrm{End}_{\mathcal{A}}(\mathfrak{X}_{\mathcal{A}}) | T \text{ has an adjoint with respect to } \langle\cdot,\cdot\rangle_{\mathcal{A}}\}. \tag{6.1}$$

We can also define, for each $x, y \in \mathfrak{X}_{\mathcal{A}}$, the “rank one” operators

$$\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_{\mathcal{A}}, \quad z \in \mathfrak{X}_{\mathcal{A}} \tag{6.2}$$

and then consider the “finite rank operators”

$$\mathfrak{F}(\mathfrak{X}_{\mathcal{A}}) = \mathsf{C}\text{-span}\{\Theta_{x,y} | x, y \in \mathfrak{X}_{\mathcal{A}}\}. \tag{6.3}$$

A simple computation shows that $\Theta_{y,x}$ is an adjoint for $\Theta_{x,y}$ and hence $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}}) \subseteq \mathfrak{B}(\mathfrak{X}_{\mathcal{A}})$. We can then regard $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$ as a *-algebra by setting $\Theta^*_{x,y} = \Theta_{y,x}$. It is easy to check that $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$ is a two-sided ideal in $\mathfrak{B}(\mathfrak{X}_{\mathcal{A}})$. Note that if $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ is a non-degenerate $\mathcal{A}$-valued inner product, then $\mathfrak{B}(\mathfrak{X}_{\mathcal{A}})$ is also a *-algebra and in this case $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$ is actually a two-sided *-ideal of $\mathfrak{B}(\mathfrak{X}_{\mathcal{A}})$. The relevance of $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$ for formal Morita equivalence is illustrated by the following proposition.

Proposition 6.1. *Suppose ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is a ($\mathcal{B}$-$\mathcal{A}$)-equivalence bimodule and that $\mathcal{B}$ has an approximate identity. Then $\mathcal{B} \cong \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$ via $\mathsf{L}_{\mathcal{B}}$.*

Proof. We know that $\mathsf{L}_{\mathcal{B}}(\mathcal{B}) \subseteq \mathfrak{B}(\mathfrak{X}_{\mathcal{A}})$ and that $\mathsf{L}_{\mathcal{B}} : \mathcal{B} \to \mathfrak{B}(\mathfrak{X}_{\mathcal{A}})$ is a *-homomorphism such that $\mathsf{L}_{\mathcal{B}}(B^*)$ is an adjoint of $\mathsf{L}_{\mathcal{B}}(B)$. Note that $\mathsf{L}_{\mathcal{B}}({}_{\mathcal{B}}\langle x, y\rangle)(z) = \langle x, y\rangle_{\mathcal{B}} \cdot z = x \cdot \langle y, z\rangle_{\mathcal{A}} = \Theta_{x,y}(z)$ and hence $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}}) \subset \mathsf{L}_{\mathcal{B}}(\mathcal{B})$. But since ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ is full, it then follows that $\mathsf{L}_{\mathcal{B}}(\mathcal{B}) = \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$. It is also easy to check that $\mathsf{L}_{\mathcal{B}} : \mathcal{B} \to \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$ is a *-homomorphism. Finally, injectivity of $\mathsf{L}_{\mathcal{B}}$ follows from Lemma 5.15. □

We also observe that the proof of Proposition 6.1 only assumed that the bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies (E1)–(E3) (not necessarily (E4)). Note that if we consider the bimodule ${}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\mathfrak{X}_{\mathcal{A}}$, with $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$-valued inner product given by

$${}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\langle x, y\rangle = \mathsf{L}_{\mathcal{B}}({}_{\mathcal{B}}\langle x, y\rangle) = \Theta_{x,y},$$

then ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \cong {}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\mathfrak{X}_{\mathcal{A}}$ as equivalence bimodules. So, given a *-algebra $\mathcal{A}$, a natural way to search for *-algebras (formally) Morita equivalent to it is by considering right $\mathcal{A}$-modules $\mathfrak{X}_{\mathcal{A}}$ endowed with a full positive semi-definite $\mathcal{A}$-valued inner product, and computing the corresponding *-algebras $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$. The difficulty is showing that the formula

$${}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\langle x, y\rangle = \Theta_{x,y} \tag{6.4}$$

is such that $\Theta_{x,x} \in \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})^+$ in general. But if one manages to do that, then we have the following proposition.

Proposition 6.2. *Let $\mathfrak{X}_{\mathcal{A}}$ be a right $\mathcal{A}$-module with a full positive semi-definite $\mathcal{A}$-valued inner product. If $\Theta_{x,x} \in \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})^+ \ \forall x \in \mathfrak{X}_{\mathcal{A}}$, then ${}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\mathfrak{K}_{\mathcal{A}}$ defines a $(\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$-$\mathcal{A})$-bimodule satisfying* (E1)–(E3).

Proof. It is clear that ${}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\langle\cdot,\cdot\rangle$ as defined in (6.4) satisfies (Y1). Note that $\Theta_{x,y}^* = \Theta_{y,x}$ implies (Y2) and since $T\Theta_{x,y} = \Theta_{Tx,y}$ for all $T \in \mathfrak{B}(\mathfrak{X}_{\mathcal{A}})$, (Y3) also holds. By our hypothesis $\Theta_{x,x} \geq 0$ and fullness is immediate from the definition of $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$. So (E1) and (E2) hold. Finally, the compatibility condition (E3) is also easy to be checked. □

Property (E4) does not seem to hold in such a general setting. It will also be useful to observe the following proposition.

Proposition 6.3. *Let $\mathfrak{X}_{\mathcal{A}}$ be as in Proposition* 6.2 *and suppose $\mathcal{A} = \mathsf{C}$. Then we automatically have $\Theta_{x,x} \in \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})^+$* (*and hence the conclusion of Proposition* 6.2 *holds*). *Note also that if $\mathcal{A} = \mathsf{C} = \hat{\mathsf{C}}$ is a field, then $\Theta_{x,x} \in \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})^{++}$.*

Proof. Just note that given any $y \in \mathfrak{X}_{\mathcal{A}}$ such that $\langle y, y\rangle_{\mathcal{A}} \neq 0$ (and one can always find such a y), then we can write $\langle y, y\rangle_{\mathcal{A}}\Theta_{x,x} = \Theta_{x,y}\Theta_{y,x} \in \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})^{++}$. But since $\langle y, y\rangle_{\mathcal{A}} \in \mathsf{C}^+$, it follows that $\Theta_{x,x} \in \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})^+$. If $\mathcal{A} = \mathsf{C} = \hat{\mathsf{C}}$ is a field, then the last claim in the proposition follows from the invertibility of $\langle y, y\rangle_{\mathcal{A}}$. □

More generally, we have the following useful results

Proposition 6.4. *Let $\mathfrak{X}_{\mathcal{A}}$ be as in Proposition* 6.2. *Suppose that for any $x \in \mathfrak{X}_{\mathcal{A}}$, there exists $y_i \in \mathfrak{X}_{\mathcal{A}}$, $i = 1, \ldots, n$ such that $x \cdot (\sum_i \langle y_i, y_i\rangle_{\mathcal{A}}) = x$. Then $\Theta_{x,x} \in \mathfrak{F}(\mathfrak{X}_{\mathcal{A}})^{++}$. In particular, it follows from Proposition* 6.2 *that in this case ${}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\mathfrak{X}_{\mathcal{A}}$ is a bimodule satisfying* (E1)–(E3).

Proof. Just note that $\sum_i \Theta_{x,y_i}\Theta_{x,y_i}^* = \sum_i \Theta_{x,y_i}\Theta_{y_i,x} = \Theta_{x\cdot\sum_i\langle y_i,y_i\rangle_{\mathcal{A}},x} = \Theta_{x,x}$. □

Corollary 6.5. *If $\mathcal{A}$ is unital and if we can write $1 = \sum_i \langle y_i, y_i\rangle_{\mathcal{A}}$ for some $y_i \in \mathfrak{X}_{\mathcal{A}}$, then ${}_{\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})}\mathfrak{X}_{\mathcal{A}}$ is a bimodule satisfying* (E1)–(E3) *and* (Y4a).

Remark 6.6. *Let us remark that in the case of C^*-algebras, $\Theta_{x,x}$ is always positive. This follows from the fact that there is a very nice characterization of the positive "compact" operators on a right Hilbert $\mathcal{A}$-module $\mathfrak{X}_{\mathcal{A}}$, namely $\mathcal{K}(\mathfrak{X}_{\mathcal{A}})^+ = \{T \in \mathcal{K}(\mathfrak{X}_{\mathcal{A}}) | \langle Tx, x\rangle_{\mathcal{A}} \geq 0 \ \forall x \in \mathfrak{X}_{\mathcal{A}}\}$ and a simple computation shows that elements of the form $\Theta_{x,x}$ belong to this set. See* [64, *Section* 2.2].

We will now use some of the previous ideas to discuss examples of formally Morita equivalent *-algebras.

Suppose Λ is any set and consider the free C-module $\mathsf{C}^{(\Lambda)} = \oplus_{i\in\Lambda}\mathsf{C}$, regarded as a right C-module with full C-valued inner product given by

$$\langle v, w\rangle_{\mathsf{C}} := \sum_i \overline{v}_i w_i. \tag{6.5}$$

Let $\{e_i\}_{i\in\Lambda}$ be the canonical basis of $\mathsf{C}^{(\Lambda)}$, which is orthonormal with respect to the inner product just defined. We define $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ as in (6.3) and observe that $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ is unital if and only if Λ is a finite set. However, note that $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ always has an approximate identity. Indeed, let F be the set of all finite subsets of Λ, with the natural partial ordering by inclusion. Then for each $J \in F$, we define $E_J = \sum_{j\in J}\Theta_{e_j,e_j}$ and one can check that $\{E_J\}_{J\in F}$ is an approximate identity of $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ (with corresponding filtration given by $\mathfrak{F}(\mathsf{C}^{(\Lambda)}) = \bigcup_{J\in F}\mathsf{C}\text{-span}\{\Theta_{e_i,e_j}|i, j \in J\}$). Also note that Proposition 2.8 implies that $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ has sufficiently many positive linear functionals.

Observe that for any $i \in \Lambda$, we have $\langle e_i, e_i\rangle = 1$ and therefore by Corollary 6.5, it follows that $\mathsf{C}^{(\Lambda)}$ is a ($\mathfrak{F}(\mathsf{C}^{(\Lambda)})$-$\mathsf{C}$) bimodule satisfying (E1)–(E3) and also (X4a′) and (Y4a′). Also observe that $\mathsf{C}^{(\Lambda)} \cong \oplus_i \mathsf{C}e_i$ and it is easy to check that (P1)–(P3) hold. Finally, note that the $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$-action on $\mathsf{C}^{(\Lambda)}$ is cyclic since, if we fix e_i, for some $i \in \Lambda$, then any $v \in \mathsf{C}^{(\Lambda)}$ can be written as $v = \Theta_{v,e_i}e_i$ and hence e_i is a cyclic vector. So (Q1)–(Q3) hold and $\mathsf{C}^{(\Lambda)}$ is a ($\mathfrak{F}(\mathsf{C}^{(\Lambda)})$-$\mathsf{C}$)-equivalence bimodule.

If Λ is a finite set, say with n elements, then $\mathsf{C}^{(\Lambda)} = \mathsf{C}^n$ and $\mathfrak{F}(\mathsf{C}^n) = \mathfrak{B}(\mathsf{C}^n) = M_n(\mathsf{C})$. So it follows that C and $M_n(\mathsf{C})$ are formally Morita equivalent. We will summarize the discussion with the following proposition.

Proposition 6.7. *The free module $\mathsf{C}^{(\Lambda)} = \oplus_{i\in\Lambda}\mathsf{C}$ has a natural ($\mathfrak{F}(\mathsf{C}^{(\Lambda)})$-$\mathsf{C}$)-equivalence bimodule structure. So $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ and C are formally Morita equivalent. In particular, C and $M_n(\mathsf{C})$ are formally Morita equivalent for all positive integers n.*

Let us now discuss the situation where, instead of $\mathsf{C}^{(\Lambda)}$, we have an arbitrary pre-Hilbert space over C, denoted by $\mathfrak{H}$ (we remark that pre-Hilbert spaces do not have orthonormal bases in general, see [21] for an example, where in fact the pre-Hilbert space is even a Hilbert space over an algebraically closed field). Let us suppose in addition that $\mathsf{C} = \hat{\mathsf{C}} = \hat{\mathsf{R}}(\mathrm{i})$ is actually a field (and $\hat{\mathsf{R}}$ is an ordered field). We can regard $\mathfrak{H}$ as a right $\hat{\mathsf{C}}$-module with full positive definite $\hat{\mathsf{C}}$-valued inner product (fullness is guaranteed by the fact that $\hat{\mathsf{C}}$ is a field). Then, by Propositions 6.2 and 6.3, it follows that $\mathfrak{H}$ is a ($\mathfrak{F}(\mathfrak{H})$-$\hat{\mathsf{C}}$)-bimodule satisfying (E1)–(E3) as well as (Y4a′). Here again, Proposition 2.8 implies that $\mathfrak{F}(\mathfrak{H})$ has sufficiently many positive linear functionals. One can check that $\mathfrak{F}(\mathfrak{H})$ acts on $\mathfrak{H}$ in a cyclic way, and in fact any non-zero vector $v \in \mathfrak{H}$ is a cyclic vector for this action. Indeed, fix $v \in \mathfrak{H}$, $v \neq 0$ and pick any $w \in \mathfrak{H}$. Then the operator $T = \Theta_{w,v}/\langle v, v\rangle \in \mathfrak{F}(\mathfrak{H})$ is such that $Tv = w$. So (Q1)–(Q3) hold. Finally, property (P) follows from the even more general fact that the tensor product of two pre-Hilbert spaces over C (not necessarily a field) is well defined (see Corollary A.7 in Appendix A). We can then state the following proposition.

Proposition 6.8. *If $\hat{\mathsf{C}} = \hat{\mathsf{R}}(\mathrm{i})$, where $\hat{\mathsf{R}}$ is an ordered field, then $\hat{\mathsf{C}}$ is formally Morita equivalent to $\mathfrak{F}(\mathfrak{H})$, where $\mathfrak{H}$ is any pre-Hilbert space over $\hat{\mathsf{C}}$.*

It is interesting to note that this is a "formal" analogue of the classical result in C^*-algebras that asserts that the algebra of compact operators on any Hilbert space is Morita equivalent to $\mathbb{C}$ (see [64, Section 3.1]). In fact, Proposition 6.8 implies that the algebra of finite rank operators on any Hilbert space is formally Morita equivalent to $\mathbb{C}$.

We will now generalize Proposition 6.7 by replacing C by an arbitrary *-algebra over C (with an approximate identity). But first, we need to discuss tensor products of equivalence bimodules. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1$ and $\mathcal{B}_2$ be *-algebras. Then it is not hard to see that the analogue of Proposition 4.5 for $\mathcal{B}_1$ and $\mathcal{B}_2$ valued inner products satisfying the corresponding conditions (Y) and (Q) also holds.

Proposition 6.9. *Let* ${}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1}$ and ${}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$ *be equivalence bimodules satisfying* (P1)–(P3) *and* (Q1)–(Q3) *as well as* (X4a) *and* (Y4a). *Let* $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ *and* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = {}_{\mathcal{B}_1}\mathfrak{X}_{\mathcal{A}_1} \otimes {}_{\mathcal{B}_2}\mathfrak{X}_{\mathcal{A}_2}$. *Then* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *is an equivalence bimodule also satisfying* (P1)–(P3), (Q1)–(Q3), (X4a) *and* (Y4a).

Proof. By Proposition 4.5 and the remark above, everything is shown except for (E3). But this follows from an easy computation. □

Let now $\mathcal{A}$ be a *-algebra over C with an approximate identity. We know that ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is an equivalence bimodule satisfying (P1)–(P3), (Q1)–(Q3), (X4a) and (Y4a). It was shown earlier in this section that ${}_{\mathfrak{F}(\mathsf{C}^{(\Lambda)})}\mathsf{C}^{(\Lambda)}{}_{\mathsf{C}}$ is an equivalence bimodule also satisfying (P1)–(P3), (Q1)–(Q3), (X4a) and (Y4a). We will consider now the $(\mathfrak{F}(\mathsf{C}^{(\Lambda)}) \otimes \mathcal{A}$-$\mathsf{C} \otimes \mathcal{A})$ bimodule given by ${}_{\mathfrak{F}(\mathsf{C}^{(\Lambda)})}\mathsf{C}^{(\Lambda)}{}_{\mathsf{C}} \otimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$. Note that $\mathfrak{F}(\mathsf{C}^{(\Lambda)}) \otimes \mathcal{A} \cong \mathfrak{F}(\mathcal{A}^{(\Lambda)})$ and $\mathsf{C} \otimes \mathcal{A} \cong \mathcal{A}$, and under this identification we can write

$$
{}_{\mathfrak{F}(\mathsf{C}^{(\Lambda)})}\mathsf{C}^{(\Lambda)}{}_{\mathsf{C}} \otimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \cong {}_{\mathfrak{F}(\mathcal{A}^{(\Lambda)})}\mathcal{A}^{(\Lambda)}_{\mathcal{A}},
$$

where $\mathcal{A}_\Lambda = \oplus_{i \in \Lambda}\mathcal{A}$. By Proposition 6.9, it then follows that ${}_{\mathfrak{F}(\mathcal{A}^{(\Lambda)})}\mathcal{A}^{(\Lambda)}_{\mathcal{A}}$ is an equivalence bimodule satisfying (P1)–(P3), (Q1)–(Q3), (X4a) and (Y4a). Based on the previous discussion, we can then state the following proposition.

Proposition 6.10. *Let $\mathcal{A}$ be a *-algebra over* C, *with an approximate identity. Then $\mathcal{A}$ and $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ are formally Morita equivalent. In particular, $\mathcal{A}$ and $M_n(\mathcal{A})$ are formally Morita equivalent for all positive integers n.*

We shall now describe a more general construction of formally Morita equivalent *-algebras. This construction will enable us to recover the results on Propositions 6.7 and 6.10 and will also lead to some important generalizations.

Let $\mathcal{A}$ be a *-algebra over C with an approximate identity $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha \in I}$, and let Λ be any set. Consider again the $\mathcal{A}$-right free module $\mathcal{A}^{(\Lambda)} = \oplus_{i \in \Lambda}\mathcal{A}$, endowed with the $\mathcal{A}$-valued inner product given by $\langle w, z\rangle = \sum_i w_i^* z_i$ for $z, w \in \mathcal{A}^{(\Lambda)}$. Observe that since $\mathcal{A}$ is assumed to have an approximate identity, this inner product is full. Let us now consider the *-algebra $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$. We remark that $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ also has an approximate identity, defined as follows. If we let $F = \{\text{Finite subsets of } \Lambda\}$, then we can consider F partially ordered by inclusion and then $F \times I$ also has a natural partial order. If $i \in \Lambda$, $\alpha \in I$, let $e_{i,\alpha} \in \mathcal{A}^{(\Lambda)}$ be the element with ith component E_α and zero elsewhere. Then given $J \in F$ and $\alpha \in I$, we set

$E_{J,\alpha} = \sum_{i\in J}\Theta_{e_{i,\alpha},e_{i,\alpha}}$ and check that $\{E_{J,\alpha}\}$ is an approximate identity (with corresponding filtration given by $\bigcup_{(J,\alpha)}\mathsf{C}\text{-span}\{\Theta_{x,y}|x_i = y_i = 0 \text{ if } i \notin J, x_i, y_i \in \mathcal{A}_\alpha\}$).

Let $Q \in \mathfrak{B}(\mathcal{A}^{(\Lambda)})$ be a projection, i.e. $Q = Q^* = Q^2$. Moreover, assume that Q satisfies

$$\mathsf{C}\text{-span}\{AQB|A, B \in \mathfrak{F}(\mathcal{A}^{(\Lambda)})\} = \mathfrak{F}(\mathcal{A}^{(\Lambda)}). \tag{6.6}$$

Such a projection is called *full*. Note that $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q \subseteq \mathfrak{F}(\mathcal{A}^{(\Lambda)})$ is a *-subalgebra, since $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ is a two-sided ideal of $\mathfrak{B}(\mathcal{A}^{(\Lambda)})$. We will now investigate when $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ and $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ are formally Morita equivalent.

Let $\mathfrak{X} = \mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$. Then $\mathfrak{X}$ has a natural ($\mathfrak{F}(\mathcal{A}^{(\Lambda)})$-$Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$)-bimodule structure, with respect to left and right multiplication. We can define $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$- and $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$-valued inner products on $\mathfrak{X}$ by

$$_{\mathfrak{F}(\mathcal{A}^{(\Lambda)})}\langle AQ, BQ\rangle = AQQ^*B^* = AQB^*, \quad A, B \in \mathfrak{F}(\mathcal{A}^{(\Lambda)}), \tag{6.7}$$

which is full since Q is a full projection, and

$$\langle AQ, BQ\rangle_{Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q} = Q^*A^*BQ = QA^*BQ, \quad A, B \in \mathfrak{F}(\mathcal{A}^{(\Lambda)}), \tag{6.8}$$

which is also full, since elements of the form A^*B span $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ (since it has an approximate identity). Also note that these inner products satisfy (X4a) and (Y4a). It is easy to check that the inner products are compatible (as in (E3)) and that $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ acts on $\mathfrak{X}$ in a pseudo-cyclic way (with pseudo-cyclic vectors $\{E_{J,\alpha}Q\}$ for $\{E_{J,\alpha}\}$ the approximate identity of $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$). So properties (Q1)–(Q3) are satisfied. We shall now discuss situations where (P) also holds and $_{\mathfrak{F}(\mathcal{A}^{(\Lambda)})}\mathfrak{X}_{Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q}$ is an equivalence bimodule. To this end, let us fix $j \in \Lambda$ and define $\mathfrak{F}^{(j)}_\alpha = \{\Theta_{e_{j,\alpha},v}|v \in \mathcal{A}^{(\Lambda)}\}$ and $\mathfrak{F}^{(j)} = \bigcup_{\alpha\in I}\mathfrak{F}^{(j)}_\alpha$. Then we can write $\mathfrak{F}(\mathcal{A}^{(\Lambda)}) = \oplus_{i\in\Lambda}\mathfrak{F}^{(i)}$ and hence

$$\mathfrak{X} = \mathfrak{F}(\mathcal{A}^{(\Lambda)})Q = \bigoplus_{i\in\Lambda}\mathfrak{F}^{(i)}Q. \tag{6.9}$$

Note that $\mathfrak{F}^{(i)}Q \subseteq \mathfrak{F}^{(i)}$ and $\mathfrak{F}^{(i)}Q \perp \mathfrak{F}^{(j)}Q$ for $i \neq j$ with respect to $\langle\cdot,\cdot\rangle_{Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q}$. Moreover, this decomposition is preserved by the right action of $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ on $\mathfrak{X}$. Let us now consider the action of $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ on $\mathfrak{F}^{(i)}Q$, for a fixed $i \in \Lambda$. We observe the following lemma, which is an easy computation.

Lemma 6.11. *For all $z, w, v \in \mathcal{A}^{(\Lambda)}$, we have*

$$\Theta_{w,z}Q(Q\Theta_{z,v}Q) = \Theta_{w\langle Qz,Qz\rangle,v}Q. \tag{6.10}$$

Suppose that for each $\alpha \in I$, we can find $z_\alpha \in \mathcal{A}^{(\Lambda)}$ and $A_\alpha \in \mathcal{A}$ such that $E_\alpha A_\alpha\langle Qz_\alpha, Qz_\alpha\rangle = E_\alpha$. In the case where $\mathcal{A}$ is unital, this is satisfied by any z such that $\langle Qz, Qz\rangle \in \mathcal{A}$ is invertible. Under this assumption, it easily follows from Lemma 6.11 that the action of $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ on $\mathfrak{F}^{(i)}Q$ is pseudo-cyclic, with pseudo-cyclic vectors $\Omega^{(i)}_\alpha = \Theta_{e_{i,\alpha}A_\alpha,z_\alpha}Q$ and thus (P1)–(P3) (and therefore (P)) are fulfilled. We remark that if $\mathcal{A}$ is unital and we find $z \in \mathcal{A}^{(\Lambda)}$ with $\langle Qz, Qz\rangle$ invertible, then the action of $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ on $\mathfrak{F}^{(i)}Q$ is actually cyclic, with cyclic vector $\Omega^{(i)} = \Theta_{e_i\langle Qz,Qz\rangle^{-1},z}Q$. We summarize the discussion with the following proposition.

Proposition 6.12. *Let $\mathcal{A}$ be a* **-algebra with an approximate identity* $\{E_\alpha\}_{\alpha\in I}$. *Let* $Q \in \mathfrak{B}(\mathcal{A}^{(\Lambda)})$ *be a full projection such that for all* $\alpha \in I$, *we can find* $z_\alpha \in \mathcal{A}^{(\Lambda)}$ *and* $A_\alpha \in \mathcal{A}$ *satisfying* $E_\alpha A_\alpha \langle Qz_\alpha, Qz_\alpha\rangle = E_\alpha$. *Then* $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ *and* $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ *are formally Morita equivalent. If* $\mathcal{A}$ *is unital, it suffices to find* $z \in \mathcal{A}^{(\Lambda)}$ *such that* $\langle Qz, Qz\rangle \in \mathcal{A}$ *is invertible and the same conclusion holds.*

We observe that Proposition 6.12 provides many examples of formally Morita equivalent *-algebras. For instance, let $i \in \Lambda$ and let $Q \in \mathfrak{B}(\mathcal{A}^{(\Lambda)})$ be the projection onto the ith coordinate. Then Q is full and for any $\alpha \in I$, we can choose $z_\alpha = e_{i,\beta}$ for some $\beta > \alpha$. Then we have $E_\alpha E_\beta \langle Qz_\alpha, Qz_\alpha\rangle = E_\alpha$. Note that $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q \cong \mathcal{A}$ and hence, by Proposition 6.12, it follows that $\mathcal{A}$ and $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ are formally Morita equivalent. Thus Proposition 6.10 follows from Proposition 6.12. Note also that if $Q, P \in \mathfrak{B}(\mathcal{A}^{(\Lambda)})$ are full projections so that $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ and $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ are formally Morita equivalent, the same holding for the pair $P\mathfrak{F}(\mathcal{A}^{(\Lambda)})P$ and $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$, then since $\mathfrak{F}(\mathcal{A}^{(\Lambda)})$ acts on $\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ and $\mathfrak{F}(\mathcal{A}^{(\Lambda)})P$ in a pseudo-cyclic way, we can apply Proposition 5.6 and conclude that $Q\mathfrak{F}(\mathcal{A}^{(\Lambda)})Q$ and $P\mathfrak{F}(\mathcal{A}^{(\Lambda)})P$ are formally Morita equivalent. We will discuss this matter a little further in the end of this section. We now observe that algebras defined by star products on Poisson manifolds, $C^\infty(M)[[\lambda]]$, have the additional property that any element of the form $1 + A^*A$ is invertible. In this case, we have the following corollary.

Corollary 6.13. *Suppose* $\mathcal{A}$ *is unital and has the property that* $1 + A^*A$ *is invertible for all* $A \in \mathcal{A}$. *If* $Q = (Q_{ij}) \in M_n(\mathcal{A})$ *is a full projection such that* Q_{ij} *is invertible in* $\mathcal{A}$ *for some* i, j, *then the conclusion of Proposition* 6.12 *holds.*

We will now concentrate our discussion in the special situation $\mathcal{A} = \mathsf{C}$. In this case, even if $\langle Qz, Qz\rangle$ is not invertible, we can always choose z so that $\langle Qz, Qz\rangle \in \mathsf{R}^+$ (for Q is full). Then one can still use Lemma 6.11 to show that this is sufficient to guarantee (P) (by an argument similar to the proof of Lemma 3.1). Finally, observe that any projection of the form $Q = \sum_{j=1}^{k} \Theta_{e_{i_j}, e_{i_j}}$ is full. In particular, if $Q = \Theta_{e_i, e_i}$ for some fixed $i \in \Lambda$, then $\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q \cong \mathsf{C}^{(\Lambda)}$ and $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q \cong \mathsf{C}$. So this example recovers the result in Proposition 6.7. Note that since the (left) action of $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ on $\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$ satisfies (Q1)–(Q3) for any full projection $Q \in \mathfrak{B}(\mathsf{C}^{(\Lambda)})$, it then follows that if $Q, P \in \mathfrak{B}(\mathsf{C})$ are full projections, then one can apply Proposition 5.6 and conclude that $\overline{\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q} \otimes_{\mathfrak{F}(\mathsf{C}^{(\Lambda)})} \mathfrak{F}(\mathsf{C}^{(\Lambda)})P$ is a $(Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$-$P\mathfrak{F}(\mathsf{C}^{(\Lambda)})P)$-equivalence bimodule. Observe that $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})P$ is a $(Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$-$P\mathfrak{F}(\mathsf{C}^{(\Lambda)})P)$-bimodule with respect to left and right multiplications, and we can also endow it with $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$ and $P\mathfrak{F}(\mathsf{C}^{(\Lambda)})P$ valued inner products in a natural way ($(QAP, QBP) \mapsto QAPB^*Q$, $(QAP, QBP) \mapsto PA^*QBP$, respectively) and an easy computation shows that, in fact, $\overline{\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q} \otimes_{\mathfrak{F}(\mathsf{C}^{(\Lambda)})} \mathfrak{F}(\mathsf{C}^{(\Lambda)})P \cong Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})P$. Hence $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})P$ is a $(Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$-$P\mathfrak{F}(\mathsf{C}^{(\Lambda)})P)$-equivalence bimodule. In particular, if $P = \Theta_{e_i, e_i}$, for some $i \in \Lambda$, then $P\mathfrak{F}(\mathsf{C}^{(\Lambda)})P \cong \mathsf{C}$ and therefore C and $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$ are formally Morita equivalent for all full projections $Q \in \mathfrak{B}(\mathsf{C}^{(\Lambda)})$. We will summarize the discussion in the following proposition.

Proposition 6.14. *Let $Q \in \mathfrak{B}(\mathsf{C}^{(\Lambda)})$ be a full projection. Then $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ and $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$ are formally Morita equivalent. Furthermore, if $P \in \mathfrak{B}(\mathsf{C}^{(\Lambda)})$ is another full projection, then it follows that $P\mathfrak{F}(\mathsf{C}^{(\Lambda)})P$ and $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$ are also formally Morita equivalent, with equivalence bimodule given by $P\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$, and moreover this bimodule satisfies* (X4a′), (Y4a′) *and it is non-degenerate*.

As in the theory of C^*-algebras, we will call the *-algebras of the form $Q\mathfrak{F}(\mathsf{C}^{(\Lambda)})Q$ *full corners* of $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$. Proposition 6.14 then states that any two full corners of $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ are formally Morita equivalent.

We end this section with a few remarks about the construction of pairs of Morita equivalent algebras out of full projections, as illustrated in Proposition 6.14. For C^*-algebras, it is known that, in fact, *all* pairs of Morita equivalent algebras arise as complementary full corners of the corresponding linking algebra (see [64, Section 3.2]). The same construction actually holds for unital *-algebras over C but the extension of these ideas to non-unital situations depends on a further development of the concept of multiplier algebra in this context. The discussion of this matter will await another time.

7. Formal versus ring-theoretic Morita equivalence

Recall that we have shown in Proposition 5.4 that two isomorphic *-algebras are also (formally) Morita equivalent. This section will be devoted to showing that the converse is also true for commutative and unital *-algebras. To this end, we will explore the relationship between the notion of formal Morita equivalence and the more standard notion of Morita equivalence for unital algebras. Let us start recalling some basic notions of Morita theory for (arbitrary) unital algebras (over some fixed unital commutative ring). See [7,50] for further details.

We say that two unital algebras A and B over a ring S are *Morita equivalent* if they have equivalent categories of left modules. A *set of equivalence data* $(A, B, \mathcal{P}, \mathcal{Q}, f, g)$ (see [7, 62 pp.]) consists of unital S-algebras A and B, bimodules $_A\mathcal{P}_B$ and $_B\mathcal{Q}_A$ and bimodule isomorphisms $f : \mathcal{P} \otimes_B \mathcal{Q} \to A$ and $g : \mathcal{Q} \otimes_A \mathcal{P} \to B$ satisfying:

1. $f(p \otimes q)p' = pg(q \otimes p')$,
2. $g(q \otimes p)q' = qf(p \otimes q')$.

A set of equivalence data is also called a *Morita context*.

Remark 7.1. *It can be shown* (*see* [7, 62 pp.]) *that if f and g are surjective homomorphisms satisfying the two conditions above, then they are actually isomorphisms*.

The main theorem of Morita theory for unital algebras asserts that A and B are Morita equivalent if and only if there exists a set of equivalence data $(A, B, \mathcal{P}, \mathcal{Q}, f, g)$ as above. Moreover, if such a set of equivalence data exists, then one can actually show (see [7, pp. 62–65]) that $\mathcal{P}$ and $\mathcal{Q}$ are finitely generated projective modules with respect to A and B. Also $\mathcal{P} \cong \mathsf{Hom}_A(\mathcal{Q}, A) \cong \mathsf{Hom}_B(\mathcal{Q}, B)$ as $(A\text{-}B)$-bimodules and $\mathcal{Q} \cong \mathsf{Hom}_B(\mathcal{P}, B) \cong$

$\mathsf{Hom}_A(\mathcal{P}, A)$ as $(B\text{-}A)$-bimodules. Moreover, $A \cong \mathrm{End}_B(\mathcal{P})$, $B \cong \mathrm{End}_A(\mathcal{Q})$ and *center*(A) $\cong \mathrm{End}({}_A\mathcal{P}_B) \cong \mathrm{End}({}_B\mathcal{Q}_A) \cong$ *center*(B). The isomorphism ϕ : *center*$(A) \to$ *center*(B) is given as follows. For each $a \in$ *center*(A), we define $\phi(a)$ as the unique $b \in$ *center*(B) such that:

$$ap = pb\, \forall p \in \mathcal{P}.$$

We also have the following characterization of Morita equivalence for unital algebras (see [50, Proposition 18.33]): two unital S-algebras A and B are Morita equivalent if and only if $A \cong eM_n(B)e$ for some full idempotent $e \in M_n(B)$. We recall that $e \in M_n(B)$ is a full idempotent if $e^2 = e$ and the S-span of BeB is B (see (6.6)).

We will now show how Morita theory for unital algebras is related to formal Morita equivalence.

Proposition 7.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital *-algebras over C and suppose ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is a bimodule satisfying* (X1)–(X3), (X5), (X6) *as well as* (Y1)–(Y3), (Y5), (Y6) *and* (E3). *Then* $(\mathcal{A}, \mathcal{B}, {}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}}, {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}, f, g)$ *is a set of equivalence data, where*

$$f : {}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \to \mathcal{A}, \quad \bar{x} \otimes y \mapsto \langle x, y \rangle_{\mathcal{A}},$$
$$g : {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}} \to \mathcal{B}, \quad x \otimes \bar{y} \mapsto {}_{\mathcal{B}}\langle x, y \rangle.$$

In particular, $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent as unital algebras.

Proof. Note that conditions (1) and (2) in the definition of a set of equivalence data hold by the compatibility condition (E3). So it remains to show that f and g are bimodule isomorphisms (it is clear that they are homomorphisms). Observe that since ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ are full, it follows that f and g are surjective. The conclusion then follows from Remark 7.1. □

We remark that we did not need the positivity conditions (X4), (Y4), (P) and (Q) for this proposition. We have the following immediate corollary.

Corollary 7.3. *If $\mathcal{A}$ and $\mathcal{B}$ are unital *-algebras over C which are formally Morita equivalent, then they are also Morita equivalent as unital C-algebras.*

Corollary 7.4. *It follows from the discussion after Remark* 7.1 *that if ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is a* $(\mathcal{B}\text{-}\mathcal{A})$-*bimodule as in Proposition* 7.2 (*in particular, if ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is an equivalence bimodule*), *then*

1. $\mathcal{B} \cong \mathfrak{F}(\mathfrak{X}_{\mathcal{A}}) \cong \mathrm{End}_{\mathcal{A}}({}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}})$,
2. $\mathcal{A} \cong \mathfrak{F}(\overline{\mathfrak{X}_{\mathcal{B}}}) \cong \mathrm{End}_{\mathcal{B}}({}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}})$,
3. *center*$(\mathcal{A}) \cong$ *center*$(\mathcal{B}) \cong \mathrm{End}({}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}})$ as C-*algebras*,
4. *There exists a full idempotent $e \in M_n(\mathcal{B})$ such that $\mathcal{A} \cong eM_n(\mathcal{B})e$.*

Note that if $\mathcal{A}$ is a *-algebra then *center*$(\mathcal{A})$ is also a *-algebra. We will now show that if $\mathcal{A}$ and $\mathcal{B}$ are unital *-algebras such that there exists a bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ as in Proposition

7.2, then *center*$(\mathcal{A}) \cong$ *center*$(\mathcal{B})$ as *-algebras. As we saw, there is an algebra isomorphism ϕ : *center*$(\mathcal{A}) \to$ *center*$(\mathcal{B})$ defined by the condition $x\mathsf{R}_{\mathcal{A}}(A) = \mathsf{L}_{\mathcal{B}}(\phi(A))x\ \forall x \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. But observe that if $T = \mathsf{L}_{\mathcal{B}}(B) = \mathsf{R}_{\mathcal{A}}(A) \in \mathrm{End}({}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}})$ (that is, T is left $\mathcal{B}$-linear and right $\mathcal{A}$-linear), we can define two adjoints for T : $T^{*_A} = \mathsf{R}_{\mathcal{A}}(A^*)$, which satisfies ${}_{\mathcal{B}}\langle Tx, y\rangle = {}_{\mathcal{B}}\langle x, T^{*_A}y\rangle$ for all $x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ or $T^{*_B} = \mathsf{L}_{\mathcal{B}}(B^*)$, which similarly satisfies $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^{*_B}y\rangle_{\mathcal{A}}$ for all $x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$.

Lemma 7.5. *Let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be a $(\mathcal{B}$-$\mathcal{A})$-bimodule satisfying* (X1)–(X3), (X5), (X6) *as well as* (Y1)–(Y3), (Y5), (Y6) *and* (E3). *Then if $T \in \mathrm{End}({}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}})$, we have $T^{*_A} = T^{*_B}$.*

Proof. Suppose $A \in$ *center*$(\mathcal{A})$. Then $A^* \in$ *center*$(\mathcal{A})$ and hence we can consider $B' = \phi(A^*)$ such that $\mathsf{R}_{\mathcal{A}}(A^*) = \mathsf{L}_{\mathcal{B}}(B')$. So it follows that ${}_{\mathcal{B}}\langle Tx, y\rangle = {}_{\mathcal{B}}\langle \mathsf{L}_{\mathcal{B}}(B)x, y\rangle = B\,{}_{\mathcal{B}}\langle x, y\rangle$. But we also have that

$$ {}_{\mathcal{B}}\langle Tx, y\rangle = {}_{\mathcal{B}}\langle x\mathsf{R}_{\mathcal{A}}(A), y\rangle = {}_{\mathcal{B}}\langle x, y\mathsf{R}_{\mathcal{A}}(A^*)\rangle = {}_{\mathcal{B}}\langle x, \mathsf{L}_{\mathcal{B}}(B')y\rangle = {}_{\mathcal{B}}\langle x, y\rangle(B')^*. $$

Hence, since $B, B' \in$ *center*$(\mathcal{B})$, we conclude that $B\,{}_{\mathcal{B}}\langle x, y\rangle = (B')^*\,{}_{\mathcal{B}}\langle x, y\rangle$. But since $\mathcal{B}$ is unital and ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ is full, it follows that $B = (B')^*$, or $B^* = B'$. In other words, $\phi(A^*) = B^*\ \forall A \in \mathcal{A}$. □

We then have the following immediate consequence.

Proposition 7.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital *-algebras such that there exists a bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ as in Lemma* 7.5. *Then center$(\mathcal{A})$ and center$(\mathcal{B})$ are *-isomorphic.*

Corollary 7.7. *If $\mathcal{A}$ and $\mathcal{B}$ are commutative unital *-algebras such that there exists a bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ as in Lemma* 7.5, *then they are *-isomorphic.*

Let us remark that similar (and even non-unital) results have recently been obtained by Ara in [5] (see note at the end of Section 10). For later use, we also observe the following corollary.

Corollary 7.8. *Let M, N be smooth manifolds and suppose there exists a $(C^\infty(M)$-$C^\infty(N))$-bimodule as in Lemma* 7.5 *($\mathbb{C}$-valued functions). Then M and N are diffeomorphic.*

Proof. By the previous proposition, $C^\infty(M)$ and $C^\infty(N)$ are *-isomorphic. So the algebras $C^\infty(M)_{\mathbb{R}}$ and $C^\infty(N)_{\mathbb{R}}$ are also isomorphic and hence M and N are diffeomorphic (see [12,74, Section 1.3.7]). □

We will now make some remarks concerning some of the previous results. First, it is immediate to conclude that unital *-algebras which are formally Morita equivalent have *-isomorphic centers, and hence if they are commutative, they must be *-isomorphic.

Note that Proposition 7.6 does not hold if we do not assume that both $\mathcal{A}$ and $\mathcal{B}$ are unital. Indeed, let us recall that, as we saw in the previous section, C and $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ are formally Morita equivalent and if Λ is not a finite set, then $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ is not unital. It is easy to check that, in this case, the center of $\mathfrak{F}(\mathsf{C}^{(\Lambda)})$ is zero whereas the center of C is C itself. However, generalizations of Proposition 7.6 and Corollary 7.7 to non-unital *-algebras (with approximate identities) are still possible (see [5, Theorem 4.2] and the note added at the end of Section 10).

Let us also remark that, unlike the case of C^*algebras (see [10, Section 1.8] and [4] for generalizations to the non-unital case), the converse of Corollary 7.3 does not hold for general *-algebras over C. To see that, let us start with a brief discussion about the algebra of smooth complex-valued functions on a compact real manifold. We recall that any algebra isomorphism $\Phi : C^\infty(M) \to C^\infty(M)$ is the lift of a diffeomorphism $\phi : M \to M$ (i.e., $\Phi = \phi^*$) (the proof of this result for real-valued functions on arbitrary manifolds, as found in [74, Section 1.3.7], also works for complex-valued functions on compact manifolds). We then have the following proposition.

Proposition 7.9. *Let M be a compact smooth manifold and let $C^\infty(M)$ denote the complex algebra of complex-valued smooth functions on M. Suppose $\Phi : C^\infty(M) \to C^\infty(M)$ is an algebra isomorphism. Then Φ must preserve conjugation: $\Phi(\bar{f}) = \overline{\Phi(f)}\,\forall f \in C^\infty(M)$.*

Corollary 7.10. *Suppose * is an involution on $C^\infty(M)$. Then $(C^\infty(M),^*)$ and $(C^\infty(M),^-)$ are isomorphic as *-algebras if and only if * is the complex conjugation.*

Suppose now that M is a compact real manifold admitting a non-trivial geometric involution (that is, a diffeomorphism ψ such that $\psi^2 = \mathrm{id}$, $\psi \neq \mathrm{id}$). Then we can define a *-involution on $C^\infty(M)$ by setting $f^* = \overline{(f \circ \psi)} = \overline{f} \circ \psi$. Then, by Corollary 7.10 $(C^\infty(M),^*)$ and $(C^\infty(M),^-)$ are not *-isomorphic (and hence not formally Morita equivalent by Corollary 7.7). But since $C^\infty(M)$ is Morita equivalent to itself as a unital complex algebra, it follows that the converse of Corollary 7.3 does not hold. Nevertheless, in some particular situations, something can be said about the converse of Corollary 7.3. We illustrate this fact with the following proposition.

Proposition 7.11. *Let $\mathsf{C} = \mathsf{R}(\mathrm{i})$ be such that $1 + \bar{x}x$ is always invertible in C. Suppose $\mathcal{A}$ is a unital C-algebra Morita equivalent to C. Then there exists an involution * in $\mathcal{A}$ such that C and $(\mathcal{A},^*)$ are formally Morita equivalent.*

Proof. If C and $\mathcal{A}$ are Morita equivalent, then as discussed before, there exists a full idempotent $e \in M_n(\mathsf{C})$ (not necessarily self-adjoint) so that $\mathcal{A} \cong eM_n(\mathsf{C})e$. Then it follows from [46, Theorem 26] that there exists a projection $Q \in M_n(\mathsf{C})$ (that is, $Q = Q^* = Q^2$) such that

$$QM_n(\mathsf{C}) = eM_n(\mathsf{C}),$$

and it is then easy to check that Q is full, for so is e. Moreover, it follows from [46, Theorem 15] that

$$QM_n(\mathsf{C})Q \cong eM_n(\mathsf{C})e,$$

and hence $\mathcal{A}$ is isomorphic to $QM_n(\mathsf{C})Q$ as a C-algebra. But since $QM_n(\mathsf{C})Q$ has a natural involution inherited from $M_n(\mathsf{C})$ (as a *-subalgebra), we can define an induced *-involution on $\mathcal{A}$, so that $\mathcal{A}$ and $QM_n(\mathsf{C})Q$ are *-isomorphic. But now it follows from Proposition 6.14 that $\mathcal{A} \cong QM_n(\mathsf{C})Q$ and C are formally Morita equivalent. $\square$

The hypothesis about $1 + \bar{x}x$ being invertible is needed for [46, Theorem 26]. Whenever $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and R is an ordered field, this is satisfied. This condition also holds for $\mathsf{C} = \mathbb{C}[[\lambda]]$. For an arbitrary unital *-algebra $\mathcal{A}$ over C, with the additional requirement that $1 + A^*A$ is invertible for all $A \in \mathcal{A}$, one can show by the same argument as in the proof of Proposition 7.11 that if $\mathcal{B}$ is another unital C-algebra Morita equivalent to $\mathcal{A}$, then we can define an involution on $\mathcal{B}$ so that there exists a ($\mathcal{B}$-$\mathcal{A}$)-bimodule satisfying (E1)–(E3).

8. Deformations of *-algebras and classical limit of *-representations

In order to make the general notion of algebraic Rieffel induction and formal Morita equivalence of *-algebras over ordered rings available for more concrete physical situations like deformation quantization we shall now investigate deformations of *-algebras and their bimodules.

Before we discuss some basic definitions and notations on *-algebra deformations, we recall that for an ordered ring R the corresponding ring of formal power series $\mathsf{R}[[\lambda]]$ is again ordered in a canonical way as we have seen for $\mathbb{R}[[\lambda]]$ in Section 2: a formal power series $a = \sum_{r=r_0}^{\infty} \lambda^r a_r \in \mathsf{R}[[\lambda]]$ is defined to be positive if $a_{r_0} > 0$. In the following we shall always use this ring ordering of $\mathsf{R}[[\lambda]]$. Moreover, we define the classical limit map $\mathfrak{C} : \mathsf{R}[[\lambda]] \to \mathsf{R}$ by taking the order zero part, i.e. $\mathfrak{C} : a \mapsto a_0$, and use $\mathfrak{C}$ similar for $\mathsf{C}[[\lambda]]$. Then $\mathfrak{C}$ is a homomorphism of ordered rings.

Now let $\mathcal{A}$ be a *-algebra over C. Then $\mathcal{A}[[\lambda]]$ is a $\mathsf{C}[[\lambda]]$-module and extending the product $\mathsf{C}[[\lambda]]$-bilinearly and the *-involution $\mathsf{C}[[\lambda]]$-antilinearly to $\mathcal{A}[[\lambda]]$ we obtain a *-algebra structure for $\mathcal{A}[[\lambda]]$ viewed as an algebra over $\mathsf{C}[[\lambda]]$. We shall refer to this *-algebra structure as 'classical' and denote the product sometimes by $\mu_0(A, B) := AB$ and the *-involution by $I_0(A) := A^*$. Then a *formal associative deformation* μ of μ_0 in the sense of Gerstenhaber [42] is a formal series $\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r$ of bilinear maps such that $(\mathcal{A}[[\lambda]], \mu)$ becomes an associative $\mathsf{C}[[\lambda]]$-algebra. A **-algebra deformation* (μ, I) of $\mathcal{A}$ is a formal associative deformation μ of $\mathcal{A}$ together with a formal series $I = \sum_{r=0}^{\infty} \lambda^r I_r$ of antilinear maps $I_r : \mathcal{A} \to \mathcal{A}$ such that I is a *-involution for the product μ, i.e. $(\mathcal{A}[[\lambda]], \mu, I)$ becomes a *-algebra over $\mathsf{C}[[\lambda]]$ such that the *classical limits* of the product μ and the *-involution I coincide with the original product μ_0 and the original *-involution I_0, respectively, see [23] for a further discussion. We shall sometimes denote the deformed product by $A \star B = \mu(A, B)$ and the deformed involution by $A^{\boldsymbol{*}} = I(A)$, and denote the classical limits again by $\mathfrak{C}\mu = \mu_0$ and $\mathfrak{C}I = I_0$. Then $\mathfrak{C} : \mathcal{A}[[\lambda]] \to \mathcal{A}$ becomes a C-linear *-homomorphism.

We shall now examine the deformed *-algebra structure more closely. First we recall the well-known fact that if V, W are C-modules and $\boldsymbol{\Phi} : V[[\lambda]] \to W[[\lambda]]$ is a $\mathsf{C}[[\lambda]]$-linear map then $\boldsymbol{\Phi}$ is actually of the form $\boldsymbol{\Phi} = \sum_{r=0}^{\infty}\lambda^r \boldsymbol{\Phi}_r$ with $\boldsymbol{\Phi}_r : V \to W$ being C-linear maps, and an analogous statement holds for multilinear maps as well, see, e.g., [33, Proposition 2.1]. In this case we shall call $\Phi = \boldsymbol{\Phi}_0 = \mathfrak{C}\boldsymbol{\Phi}$ again the classical limit of $\boldsymbol{\Phi}$. Thus let a *-algebra deformation (μ, I) of $\mathcal{A}$ be given and consider a positive $\mathsf{C}[[\lambda]]$-linear functional $\omega : \mathcal{A}[[\lambda]] \to \mathsf{C}[[\lambda]]$ which can thus be written as $\omega = \sum_{r=0}^{\infty}\lambda^r \omega_r$ with C-linear functionals $\omega_r : \mathcal{A} \to \mathsf{C}$. From $\omega(A^* \star A) \geq 0$ and the definition of the ordering of $\mathsf{R}[[\lambda]]$ it follows immediately that the *classical limit* $\omega_0 = \mathfrak{C}\omega$ of ω is a positive C-linear functional of the classical *-algebra $\mathcal{A}$, see also [21, Lemma 6] for a formulation in the context of deformation quantization. This raises the question of whether every classically positive linear functional ω_0 is automatically positive for the deformed *-algebra. A simple example shows that in general this is *not* the case [21, Section 2] and thus one is led to the refined question of whether one can *deform* a classically positive C-linear functional ω_0 into a positive $\mathsf{C}[[\lambda]]$-linear functional ω of the deformed *-algebra by adding appropriate higher order terms, i.e. $\omega = \sum_{r=0}^{\infty}\lambda^r \omega_r$. If this is possible for *all* classically positive linear functionals then we shall call the *-algebra deformation (μ, I) a *positive deformation* of $\mathcal{A}$. It turns out that many interesting examples and in particular all Hermitian star products on symplectic manifolds have this property [23, Proposition 5.1]. Moreover, the important property of having sufficiently many positive linear functionals is preserved under positive deformations [23, Proposition 4.2].

Let us recall the definition of the λ-adic order and the λ-adic absolute value: let V be a C-module and consider $v = \sum_{r=0}^{\infty}\lambda^r v_r \in V[[\lambda]]$. Then the order of v is defined by $\mathrm{o}(v) = \min\{r | v_r \neq 0\}$, where we set $\mathrm{o}(0) = +\infty$, and the absolute value of v is defined by $\varphi(v) = 2^{-\mathrm{o}(v)}$. Then $d(v, w) = \varphi(v - w)$ defines an ultra-metric for $v, w \in V[[\lambda]]$ and $V[[\lambda]]$ is a complete metric space. The corresponding topology is called the λ-adic topology and clearly $V[[\lambda]]$ is a topological module over the topological ring $\mathsf{C}[[\lambda]]$, see, e.g., [21,75] for a more extensive treatment of the λ-adic and related topologies. The way we shall use these topological aspects of formal power series is that we may use some less restrictive axioms by replacing various 'fullness conditions' by their 'dense' analogues. Then the automatic continuity of $\mathsf{C}[[\lambda]]$-linear maps (see above) ensures that the corresponding constructions still work. In particular we shall need the following definition of a *topological approximate identity*: Let $\boldsymbol{\mathcal{A}}_\alpha \subseteq \boldsymbol{\mathcal{A}}_\beta$ for $\alpha \leq \beta \in I$ be a system of directed $\mathsf{C}[[\lambda]]$-submodules of $\boldsymbol{\mathcal{A}} = \mathcal{A}[[\lambda]]$ such that $\bigcup_{\alpha\in I}\boldsymbol{\mathcal{A}}_\alpha$ is *dense* in $\boldsymbol{\mathcal{A}}$ and let $\boldsymbol{E}_\alpha \in \boldsymbol{\mathcal{A}}$ be elements such that $\boldsymbol{E}_\alpha \star \boldsymbol{E}_\beta = \boldsymbol{E}_\alpha = \boldsymbol{E}_\beta \star \boldsymbol{E}_\alpha$ for $\alpha < \beta$, $\boldsymbol{E}_\alpha^* = \boldsymbol{E}_\alpha$, and for all $A \in \boldsymbol{\mathcal{A}}_\alpha$ one has $\boldsymbol{E}_\alpha \star A = A = A \star \boldsymbol{E}_\alpha$. Then $\{\boldsymbol{\mathcal{A}}_\alpha, \boldsymbol{E}_\alpha\}_{\alpha\in I}$ is called a topological approximate identity. The classical limit of a topological approximate identity yields an approximate identity.

Lemma 8.1. *Let $\mathcal{A}$ be a *-algebra over C and let $(\boldsymbol{\mathcal{A}} = \mathcal{A}[[\lambda]], \mu, I)$ be a *-algebra deformation of $\mathcal{A}$ admitting a topological approximate identity $\{\boldsymbol{\mathcal{A}}_\alpha, \boldsymbol{E}_\alpha\}_{\alpha\in I}$. Then the classical limit $\mathcal{A}_\alpha := \mathfrak{C}(\boldsymbol{\mathcal{A}}_\alpha) = \boldsymbol{\mathcal{A}}_\alpha \cap \mathcal{A}$ and $E_\alpha := \mathfrak{C}(\boldsymbol{E}_\alpha)$ defines an approximate identity $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha\in I}$ of $\mathcal{A}$. In particular, if $\boldsymbol{\mathcal{A}}$ has a unit then the classical limit $\mathfrak{C}\mathbf{1} = 1$ is a unit for $\mathcal{A}$.*

Proof. It is clear that $\{\mathcal{A}_\alpha\}_{\alpha\in I}$ defines a directed filtered system of submodules of $\mathcal{A}$. Now let $A \in \mathcal{A}$ be given then we find a sequence $A_n \in \bigcup_\alpha \boldsymbol{\mathcal{A}}_\alpha$ converging to A in the λ-adic topology. But $A_n = \sum_{r=0}^{\infty}\lambda^r A_n^{(r)}$ can only converge to A if there exists a N such that for all $n \geq N$ we have $A_n^{(0)} = A$. Since on the other hand $A_n \in \boldsymbol{\mathcal{A}}_{\alpha_n}$ for some α_n we conclude $A \in \mathcal{A}_{\alpha_n}$ for $n \geq N$ whence $\bigcup_\alpha \mathcal{A}_\alpha = \mathcal{A}$ is shown. It remains to show the defining properties of the E_α which is straightforward. □

Note that some $\mathcal{A}_\alpha$ might be trivial and some E_α might be 0. Note also that star products (with bidifferential operators vanishing on the constants) provide an example, where one also can 'quantize' an approximate identity, see [75]. Let M be a manifold and let $\{O_n\}_{n\in\mathbb{N}}$ be open subsets of M such that $O_n^{\mathrm{cl}} \subset O_{n+1}$, O_n^{cl} is compact, and $\bigcup_n O_n = M$. Moreover, choose $\chi_n \in C_0^\infty(M)$ such that $\mathrm{supp}\chi_n \subseteq O_{n+1}$ and $\chi_n|_{O_n^{\mathrm{cl}}} = 1$. Then $\{C_0^\infty(O_n)[[\lambda]], \chi_n\}_{n\in\mathbb{N}}$ is a topological approximate identity for any (local) star product on M (for any Poisson structure) and the classical limit is $\{C_0^\infty(O_n), \chi_n\}$. Note that if M is non-compact this is only a topological approximate identity since $\bigcup_n C_0^\infty(O_n)[[\lambda]] \neq C_0^\infty(M)[[\lambda]]$. Furthermore, we notice that a topological approximate identity is sufficient for Proposition 2.8.

In order to discuss the classical limit of *-representations and bimodules of deformed *-algebras we first have to consider the classical limit of pre-Hilbert spaces. Let $\boldsymbol{\mathfrak{H}}$ be a pre-Hilbert space over $\mathsf{C}[[\lambda]]$ then we want to define its 'classical limit' in order to get a pre-Hilbert space over C. The first guess might be $\boldsymbol{\mathfrak{H}}/\lambda\boldsymbol{\mathfrak{H}}$ but it turns out that this space is sometimes still too big and does not necessarily allow for a reasonable C-valued Hermitian product.

Lemma 8.2. *Let $\boldsymbol{\mathfrak{H}}$ be a pre-Hilbert space over $\mathsf{C}[[\lambda]]$. Then $\{\phi \in \boldsymbol{\mathfrak{H}} | \mathfrak{C}\langle\phi,\phi\rangle = 0\}$ coincides with the $\mathsf{C}[[\lambda]]$-submodule $\boldsymbol{\mathfrak{H}}_L := \{\phi \in \boldsymbol{\mathfrak{H}} | \mathfrak{C}\langle\phi,\psi\rangle = 0\, \forall\psi \in \boldsymbol{\mathfrak{H}}\}$ and clearly $\lambda\boldsymbol{\mathfrak{H}} \subseteq \boldsymbol{\mathfrak{H}}_L$. Thus the quotient $\mathfrak{C}\boldsymbol{\mathfrak{H}} := \boldsymbol{\mathfrak{H}}/\boldsymbol{\mathfrak{H}}_L$ is canonically a pre-Hilbert space over C with the Hermitian product*

$$\langle\mathfrak{C}\phi, \mathfrak{C}\psi\rangle := \mathfrak{C}(\langle\phi,\psi\rangle), \tag{8.1}$$

where $\mathfrak{C} : \boldsymbol{\mathfrak{H}} \to \mathfrak{C}\boldsymbol{\mathfrak{H}}$ denotes the projection.

Proof. Let ϕ satisfy $\mathfrak{C}\langle\phi,\phi\rangle = 0$ and $\psi \in \boldsymbol{\mathfrak{H}}$. Then $\langle\phi,\psi\rangle\langle\psi,\phi\rangle \leq \langle\phi,\phi\rangle\langle\psi,\psi\rangle$ shows that $\mathfrak{C}(\langle\phi,\psi\rangle) = 0$ since $\langle\psi,\psi\rangle$ has non-negative λ-adic order which proves the first part since the other inclusion is trivial. The other statements are straightforward. □

We shall call $\mathfrak{H} = \mathfrak{C}\boldsymbol{\mathfrak{H}}$ the *classical limit* of $\boldsymbol{\mathfrak{H}}$. Observe also the useful formula

$$\mathfrak{C}(z\phi + w\psi) = \mathfrak{C}(z)\mathfrak{C}(\phi) + \mathfrak{C}(w)\mathfrak{C}(\psi) \tag{8.2}$$

for $z, w \in \mathsf{C}[[\lambda]]$ and $\phi, \psi \in \boldsymbol{\mathfrak{H}}$. Since the higher powers of λ act trivially on the $\mathsf{C}[[\lambda]]$-module $\mathfrak{C}\boldsymbol{\mathfrak{H}}$ it is reasonable to consider $\mathfrak{C}\boldsymbol{\mathfrak{H}}$ only as C-module. If $\mathfrak{H}$ is a pre-Hilbert space over C then $\boldsymbol{\mathfrak{H}} = \mathfrak{H}[[\lambda]]$ becomes a pre-Hilbert space over $\mathsf{C}[[\lambda]]$ by extending the Hermitian product $\mathsf{C}[[\lambda]]$-(anti)linearly. In this case clearly $\mathfrak{C}\boldsymbol{\mathfrak{H}} \cong \mathfrak{H}$ in a canonical way. But note that $\mathfrak{C}$ is defined for *all* pre-Hilbert spaces over $\mathsf{C}[[\lambda]]$ which are not necessarily of that form. Note also that it may happen that $\mathfrak{C}\boldsymbol{\mathfrak{H}} = \{0\}$ even if $\boldsymbol{\mathfrak{H}} \neq \{0\}$ (just rescale the

Hermitian product by λ). Next we shall consider the morphisms of pre-Hilbert spaces and their classical limit.

Lemma 8.3. *Let* $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$ *be pre-Hilbert spaces over* $\mathsf{C}[[\lambda]]$ *and let* $A, A' \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$, $B \in \mathfrak{B}(\mathfrak{H}_2, \mathfrak{H}_3)$ *and* $z, w \in \mathsf{C}[[\lambda]]$.

1. $A(\mathfrak{H}_{1L}) \subseteq \mathfrak{H}_{2L}$ *whence* $\mathfrak{C}A : \mathfrak{C}\mathfrak{H}_1 \to \mathfrak{C}\mathfrak{H}_2$ *defined by*
$$\mathfrak{C}A(\mathfrak{C}\phi) := \mathfrak{C}(A\phi) \tag{8.3}$$
is well-defined and C*-linear*.
2. $\mathfrak{C}(zA + wA') = \mathfrak{C}(z)\mathfrak{C}(A) + \mathfrak{C}(w)\mathfrak{C}(A')$, $\mathfrak{C}A \in \mathfrak{B}(\mathfrak{C}\mathfrak{H}_1, \mathfrak{C}\mathfrak{H}_2)$ *with* $(\mathfrak{C}A)^* = \mathfrak{C}(A^*)$, *and* $\mathfrak{C}(BA) = (\mathfrak{C}B)(\mathfrak{C}A)$.

Proof. Let $\phi \in \mathfrak{H}_{1L}$ then $\mathfrak{C}\langle A\phi, A\phi\rangle = \mathfrak{C}\langle A^*A\phi, \phi\rangle = 0$ according to Lemma 8.2. Thus $A\phi \in \mathfrak{H}_{2L}$ and $\mathfrak{C}A$ is a well-defined C-linear map. The second part is an easy computation. □

In other words we obtain a *functor* $\mathfrak{C}$ from the category of pre-Hilbert spaces over $\mathsf{C}[[\lambda]]$ into the category of pre-Hilbert spaces over C. Note that the fact that $\mathsf{R}[[\lambda]]$ is ordered was crucial for this construction of $\mathfrak{C}$. We shall refer to $\mathfrak{C}$ as the *classical limit functor*.

Now we shall investigate the classical limit of *-representations of deformed algebras. Let $\mathcal{A}$ be a *-algebra over C and let $(\boldsymbol{\mathcal{A}} = \mathcal{A}[[\lambda]], \mu, I)$ be a *-algebra deformation of $\mathcal{A}$. For a *-representation of $\boldsymbol{\mathcal{A}}$ we obtain the following lemma.

Lemma 8.4. *Let* $(\boldsymbol{\mathcal{A}}, \mu, I)$ *be a* **-algebra deformation of a* **-algebra* $\mathcal{A}$ *over* C *and let* $\boldsymbol{\pi} : \boldsymbol{\mathcal{A}} \to \mathfrak{B}(\mathfrak{H})$ *be a* **-representation of* $\boldsymbol{\mathcal{A}}$ *on a pre-Hilbert space* $\mathfrak{H}$ *over* $\mathsf{C}[[\lambda]]$. *Then* $\pi = \mathfrak{C}\boldsymbol{\pi} : \mathcal{A} = \mathfrak{C}\boldsymbol{\mathcal{A}} \to \mathfrak{B}(\mathfrak{C}\mathfrak{H})$
$$(\mathfrak{C}\boldsymbol{\pi}(\mathfrak{C}A))\mathfrak{C}\phi := \mathfrak{C}(\boldsymbol{\pi}(A)\phi) \tag{8.4}$$
defines a **-representation of* $\mathcal{A}$ *on* $\mathfrak{C}\mathfrak{H}$.

Proof. The well-definedness is shown analogously to the last lemma and the *-representation properties are a straightforward computation. □

Let us now discuss how additional properties of a *-representation as mentioned in Section 2 behave under the classical limit. First it is clear that even if $\boldsymbol{\pi}$ is faithful then $\mathfrak{C}\boldsymbol{\pi}$ need not to be faithful at all. While it is not clear in general whether the classical limit of a non-degenerate *-representation is again non-degenerate, this is certainly true for strong non-degeneracy: if for all $\phi \in \mathfrak{H}$ we find $A_i \in \boldsymbol{\mathcal{A}}$ and $\psi_i \in \mathfrak{H}$ such that $\phi = \sum_i \boldsymbol{\pi}(A_i)\psi_i$ then $\mathfrak{C}\phi = \sum_i \mathfrak{C}\boldsymbol{\pi}(\mathfrak{C}A_i)\mathfrak{C}\psi_i$ shows that $\mathfrak{C}\boldsymbol{\pi}$ is strongly non-degenerate. Now assume $\boldsymbol{\pi}$ is pseudo-cyclic with filtration $\{\mathfrak{H}_\alpha\}_{\alpha\in I}$ and pseudo-cyclic vectors $\boldsymbol{\Omega}_\alpha$. Then define $\mathfrak{H}_\alpha := \mathfrak{C}\mathfrak{H}_\alpha$ and $\Omega_\alpha := \mathfrak{C}\boldsymbol{\Omega}_\alpha$. Then it is easy to check that $\{\mathfrak{H}_\alpha\}_{\alpha\in I}$ defines a filtration of $\mathfrak{H} = \mathfrak{C}\mathfrak{H}$ and Ω_α are pseudo-cyclic vectors for $\pi = \mathfrak{C}\boldsymbol{\pi}$. If $\boldsymbol{\pi}$ is compatible with the filtration $\{\mathfrak{H}_\alpha\}_{\alpha\in I}$ then π is compatible with the filtration $\{\mathfrak{H}_\alpha\}_{\alpha\in I}$. Let us finally consider an isometric intertwiner $\boldsymbol{T} : \mathfrak{H}_1 \to \mathfrak{H}_2$ for two *-representations $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ of $\boldsymbol{\mathcal{A}}$. Then the map $T := \mathfrak{C}\boldsymbol{T} : \mathfrak{C}\mathfrak{H}_1 \to \mathfrak{C}\mathfrak{H}_2$ defined by $\mathfrak{C}\boldsymbol{T}(\mathfrak{C}\phi) := \mathfrak{C}(\boldsymbol{T}\phi)$ is well-defined since $\boldsymbol{T}$ is isometric. Moreover, T is linear, still isometric, and obviously an intertwiner for $\mathfrak{C}\boldsymbol{\pi}_1$ and $\mathfrak{C}\boldsymbol{\pi}_2$. If $\boldsymbol{T}$ is even unitary then T is also unitary with inverse $T^{-1} = \mathfrak{C}(\boldsymbol{T}^{-1})$. Adjointable intertwiners are already covered by Lemma 8.3. We summarize these results in the following proposition.

Proposition 8.5. *Let $(\boldsymbol{\mathcal{A}} = \mathcal{A}[[\lambda]], \mu, I)$ be a *-algebra deformation of a *-algebra $\mathcal{A}$ over C. Then taking the classical limit of *-representations yields a functor*

$$\mathfrak{C} : {}^*\text{-rep}(\boldsymbol{\mathcal{A}}) \to {}^*\text{-rep}(\mathcal{A}), \tag{8.5}$$

which maps strongly non-degenerate, filtered, and pseudo-cyclic *-representations to strongly non-degenerate, filtered, and pseudo-cyclic *-representations, respectively.

Remark 8.6. *Note that this functor is not of the type of those functors obtained by algebraic Rieffel induction since here we consider a functor between categories of *-representations of *-algebras over different rings.*

9. Classical limit and deformation of bimodules

With the set-up of the previous section, we now turn to the question of the classical limit and deformation of bimodules. Let $(\boldsymbol{\mathcal{A}} = \mathcal{A}[[\lambda]], \mu_{\mathcal{A}}, I_{\mathcal{A}})$ and $(\boldsymbol{\mathcal{B}} = \mathcal{B}[[\lambda]], \mu_{\mathcal{B}}, I_{\mathcal{B}})$ be *-algebra deformations of *-algebras $\mathcal{A}$ and $\mathcal{B}$ over C. We consider a $\mathsf{C}[[\lambda]]$-module $\boldsymbol{\mathfrak{X}}$ which is equipped with a $(\boldsymbol{\mathcal{B}}$-$\boldsymbol{\mathcal{A}})$-bimodule structure and a $\boldsymbol{\mathcal{A}}$-valued inner product, then the first question is how to define the classical limit of ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$. To this end we shall first discuss the general case and specialize to more concrete cases afterwards. We use the $\boldsymbol{\mathcal{A}}$-valued inner product to define the classical limit of ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ similarly to the classical limit of pre-Hilbert spaces. Consider the $\mathsf{C}[[\lambda]]$-submodule

$$\boldsymbol{\mathfrak{X}}_L := \{\boldsymbol{x} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} | \forall \boldsymbol{y} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} : \mathfrak{C}\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}} = 0\}. \tag{9.1}$$

Then clearly $\lambda\, {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} \subseteq {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$. We are thus able to define the *classical limit* of ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ as the quotient

$$\mathfrak{X} = \mathfrak{C}\, {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} := {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} / \boldsymbol{\mathfrak{X}}_L, \tag{9.2}$$

and denote by $x = \mathfrak{C}\boldsymbol{x} \in \mathfrak{X}$ the equivalence class of $\boldsymbol{x} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$. Though $\mathfrak{X}$ is in principle a $\mathsf{C}[[\lambda]]$-module, all higher powers of λ act trivially on $\mathfrak{X}$ whence we regard $\mathfrak{X}$ as C-module only. We shall now prove that all relevant structures pass to the classical limit. First notice that the $\boldsymbol{\mathcal{B}}$-left action $\mathsf{L}_{\boldsymbol{\mathcal{B}}}$ as well as the $\boldsymbol{\mathcal{A}}$-right action $\mathsf{R}_{\boldsymbol{\mathcal{A}}}$ pass to the quotient due to (X5) and (X3), respectively. Moreover, it is clear they yield left and right actions of the classical limits. Thus we can define a $(\mathcal{B}$-$\mathcal{A})$-bimodule structure on $\mathfrak{X}$ by setting

$$\mathsf{L}_{\mathcal{B}}(B)(\mathfrak{C}\boldsymbol{x}) := \mathfrak{C}(\mathsf{L}_{\mathcal{B}}(B)\boldsymbol{x}) \quad \text{and} \quad (\mathfrak{C}\boldsymbol{x})\mathsf{R}_{\mathcal{A}}(A) := \mathfrak{C}(\boldsymbol{x}\mathsf{R}_{\mathcal{A}}(A)), \tag{9.3}$$

which gives indeed a well-defined ($\mathcal{B}$-$\mathcal{A}$)-bimodule structure on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} := \mathfrak{X}$. Next, one checks that

$$\langle \mathfrak{C}\boldsymbol{x}, \mathfrak{C}\boldsymbol{y}\rangle_{\mathcal{A}} := \mathfrak{C}(\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}}) \tag{9.4}$$

defines an $\boldsymbol{\mathcal{A}}$-valued inner product, where the well-definedness follows directly from (9.1). Note that $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ automatically satisfies the property that if $\langle \mathfrak{C}\boldsymbol{x}, \mathfrak{C}\boldsymbol{y}\rangle_{\mathcal{A}} = 0$ for all $\mathfrak{C}\boldsymbol{y}$ then $\mathfrak{C}\boldsymbol{x} = 0$. Nevertheless note, as in Remark 5.17, that it is not necessarily true that $\langle \mathfrak{C}\boldsymbol{x}, \mathfrak{C}\boldsymbol{x}\rangle_{\mathcal{A}} = 0$ implies $\mathfrak{C}\boldsymbol{x} = 0$. Moreover, the various properties of $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ are inherited by $\langle\cdot,\cdot\rangle_{\mathcal{A}}$:

Lemma 9.1. *If* $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ *satisfies* (X1)–(X3), (X4a), *and* (X5) *then* $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ *satisfies* (X1)–(X3), (X4a), *and* (X5), *respectively. If* ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *satisfies* (P1)–(P3) *then* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *also satisfies* (P1)–(P3).

Proof. The properties (X1)–(X3), (X4a), and (X5) are an easy check. Thus let us consider (P1) where we assume $\boldsymbol{\mathfrak{X}} = \oplus_{i\in I}\boldsymbol{\mathfrak{X}}^{(i)}$. Then clearly $\sum_{i\in I}\mathfrak{C}\boldsymbol{\mathfrak{X}}^{(i)}$ coincides with the whole space $\mathfrak{C}\boldsymbol{\mathfrak{X}}$ and the sum is also orthogonal. But from the above remark we conclude that the sum is also direct and hence (P1) is valid for the classical limit. Finally (P2) is obvious and (P3) follows by taking $\mathfrak{C}\boldsymbol{\Omega}_{\alpha}^{(i)}$ as pseudo-cyclic vectors for $\mathfrak{C}\boldsymbol{\mathfrak{X}}^{(i)}$. □

For the fullness condition (X6) we may even use a topological version using the λ-adic topology of $\boldsymbol{\mathcal{A}}$. We define

(tX6) $\mathsf{C}[[\lambda]]$-span $\{\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}} | \boldsymbol{x}, \boldsymbol{y} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}\}$ is λ-adically dense in $\boldsymbol{\mathcal{A}} = \mathcal{A}[[\lambda]]$,

which actually will be sufficient for our constructions. In particular, the classical limit of (tX6) yields (X6) through analogous arguments as in the proof of Lemma 8.1:

Lemma 9.2. *If the* $\boldsymbol{\mathcal{A}}$*-valued inner product* $\langle\cdot,\cdot\rangle_{\boldsymbol{\mathcal{A}}}$ *satisfies* (tX6) *then the classical limit* $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ *satisfies* (X6).

Let us finally discuss the two positivity requirements (X4) and (P), which turn out to be more involved. We have already observed that the classical limit of a positive $\mathsf{C}[[\lambda]]$-linear functional of $\boldsymbol{\mathcal{A}}$ is a positive C-linear functional of $\mathcal{A}$. On the other hand, there may be 'fewer' positive $\mathsf{C}[[\lambda]]$-linear functionals of $\boldsymbol{\mathcal{A}}$ and thus the condition $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{A}}} \in \boldsymbol{\mathcal{A}}^+$ for $\boldsymbol{x} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ would imply only a *weaker* condition in the classical limit and thus one could not necessarily guarantee $\langle \mathfrak{C}\boldsymbol{x}, \mathfrak{C}\boldsymbol{x}\rangle_{\mathcal{A}} \in \mathcal{A}^+$ for $\mathfrak{C}\boldsymbol{x} \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$, i.e. (X4) for the classical limit. Nevertheless, in the case of a *positive deformation* we have 'enough' positive linear functionals for $\boldsymbol{\mathcal{A}}$:

Lemma 9.3. *Let* $(\boldsymbol{\mathcal{A}}, \mu_{\mathcal{A}}, I_{\mathcal{A}})$ *be a positive deformation of* $\mathcal{A}$ *and* ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *a* ($\boldsymbol{\mathcal{B}}$-$\boldsymbol{\mathcal{A}}$)*-bimodule with* $\boldsymbol{\mathcal{A}}$*-valued inner product. Then* (X4) *for* $\langle\cdot,\cdot\rangle_{\boldsymbol{\mathcal{A}}}$ *implies* (X4) *for the classical limit* $\langle\cdot,\cdot\rangle_{\mathcal{A}}$.

Proof. Let $x = \mathfrak{C}\boldsymbol{x} \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ then we have to prove $\omega_0(\langle x, x\rangle_{\mathcal{A}}) \geq 0$ for all positive linear functionals $\omega_0 : \mathcal{A} \to \mathsf{C}$. Choose a positive $\mathsf{C}[[\lambda]]$-linear functional $\omega = \sum_{r=0}^{\infty}\lambda^r\omega_r$: $\boldsymbol{\mathcal{A}} \to \mathsf{C}[[\lambda]]$ with $\mathfrak{C}\omega = \omega_0$ which exists since $\boldsymbol{\mathcal{A}}$ is a positive deformation. Then $\omega(\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{A}}})$ ≥ 0 by (X4) implies $\omega_0(\langle x, x\rangle_{\mathcal{A}}) \geq 0$. □

Concerning the property (P) we face an analogous problem as for (X4) since if $(\mathfrak{H}, \pi)$ is a *-representation of $\mathcal{A}$ which appears as classical limit of a *-representation $(\boldsymbol{\mathfrak{H}}, \boldsymbol{\pi})$ of $\boldsymbol{\mathcal{A}}$ then we can easily conclude the semi-definite positivity of the induced inner product $\langle \cdot, \cdot \rangle_{\tilde{\mathfrak{K}}}$ by taking the classical limit everywhere. The problem arises since not all *-representation of $\mathcal{A}$ have to necessarily appear as classical limit of a *-representation of $\boldsymbol{\mathcal{A}}$. Thus one is led to the question of *deformability of* **-representations* of $\mathcal{A}$ into *-representations of a given *-algebra deformation $\boldsymbol{\mathcal{A}}$ of $\mathcal{A}$. We shall not discuss this matter any further in this work but leave this as an open question for future investigations. Nevertheless, in most of our examples the property (P) follows either from (P1)–(P3), which behave well with respect to the classical limit, or can be shown directly by other techniques. Let us summarize the results so far in the following proposition.

Proposition 9.4. *Let* $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ *be* **-algebra deformations of* **-algebras* $\mathcal{A}$, $\mathcal{B}$ *over* C *and let* ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *be a* $(\boldsymbol{\mathcal{B}}$-$\boldsymbol{\mathcal{A}})$*-bimodule with a* $\boldsymbol{\mathcal{A}}$*-valued inner product* $\langle \cdot, \cdot \rangle_{\boldsymbol{\mathcal{A}}}$ *such that the properties* (X1)–(X3), (X4a), (X5), (tX6) *or* (P1)–(P3) *are satisfied. Then the classical limit* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = \mathfrak{C}\,{}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *carries a* $(\mathcal{B}$-$\mathcal{A})$*-bimodule structure and a* $\mathcal{A}$*-valued inner product satisfying* (X1)–(X3), (X4a), (X5), (X6), *or* (P1)–(P3), *respectively. If in addition* $\boldsymbol{\mathcal{A}}$ *is a positive deformation and* $\langle \cdot, \cdot \rangle_{\boldsymbol{\mathcal{A}}}$ *only satisfies* (X4) *instead of* (X4a) *then* $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ *also satisfies* (X4).

A simple computation yields the following useful relation between the functor $\mathfrak{R}_{\boldsymbol{\mathfrak{X}}}$ of algebraic Rieffel induction coming form a $(\boldsymbol{\mathcal{B}}$-$\boldsymbol{\mathcal{A}})$-bimodule and the functor $\mathfrak{R}_{\mathfrak{X}}$ of the corresponding classical limit.

Proposition 9.5. *Let* $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ *be* **-algebra deformations of* **-algebras* $\mathcal{A}$, $\mathcal{B}$ *over* C *and let* ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *be a* $(\boldsymbol{\mathcal{B}}$-$\boldsymbol{\mathcal{A}})$*-bimodule with* $\boldsymbol{\mathcal{A}}$*-valued inner product satisfying* (X1)–(X5) *and* (P). *Assume furthermore that the classical limit* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *also satisfies* (X4) *and* (P). *Then the functors* $\mathfrak{C} \circ \mathfrak{R}_{\boldsymbol{\mathfrak{X}}}$ *and* $\mathfrak{R}_{\mathfrak{X}} \circ \mathfrak{C}$ *are naturally isomorphic: for a* **-representation* $(\boldsymbol{\mathfrak{H}}, \boldsymbol{\pi})$ *of* $\boldsymbol{\mathcal{A}}$ *the map* $U : \mathfrak{C}\mathfrak{R}_{\boldsymbol{\mathfrak{X}}}(\boldsymbol{\mathfrak{H}}) \to \mathfrak{R}_{\mathfrak{X}}\mathfrak{C}(\boldsymbol{\mathfrak{H}})$ *defined for* $\boldsymbol{x} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *and* $\phi \in \boldsymbol{\mathfrak{H}}$ *by*

$$U(\mathfrak{C}([\boldsymbol{x} \otimes \phi])) := [\mathfrak{C}\boldsymbol{x} \otimes \mathfrak{C}\phi] \tag{9.5}$$

is a unitary intertwiner between $\mathfrak{C}\mathfrak{R}_{\boldsymbol{\mathfrak{X}}}(\boldsymbol{\pi})$ *and* $\mathfrak{R}_{\mathfrak{X}}\mathfrak{C}(\boldsymbol{\pi})$.

Proof. Using the present results the well-definedness of U is easily established. The rest is a simple computation. □

Let us now turn to equivalence bimodules for $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{A}}$, where we shall assume that the undeformed *-algebras $\mathcal{A}$ and $\mathcal{B}$ have an approximate identity. Then given an equivalence bimodule ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ we have in principle two ways to define the classical limit: either we use the $\boldsymbol{\mathcal{A}}$-valued inner product to define $\boldsymbol{\mathfrak{X}}_L$ or we use the $\boldsymbol{\mathcal{B}}$-valued inner product to define ${}_L\boldsymbol{\mathfrak{X}} := \{x \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} | \forall \boldsymbol{y} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} : \mathfrak{C}_{\boldsymbol{\mathcal{B}}}\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0\}$ and use the corresponding quotients as classical limit. Fortunately, both spaces coincide and we actually do not need the positivity requirements:

Lemma 9.6. *Let* $_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *be a* $(\boldsymbol{\mathcal{B}}\text{-}\boldsymbol{\mathcal{A}})$*-bimodule with* $\boldsymbol{\mathcal{A}}$*- and* $\boldsymbol{\mathcal{B}}$*-valued inner products satisfying* (X1)–(X3), (X5), (tX6) *and* (Y1)–(Y3), (Y5), (tY6), *respectively, as well as* (E3). *Then* $_L\boldsymbol{\mathfrak{X}} = \boldsymbol{\mathfrak{X}}_L$.

Proof. We can proceed almost analogously as in the proof of Proposition 5.16. First we need the following analogue of Lemma 5.15: let $B \in \mathcal{B}$ and assume $\mathsf{L}_{\boldsymbol{\mathcal{B}}}(B)\boldsymbol{x} \in \lambda_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ for all $\boldsymbol{x} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ and let $E_\alpha \in \mathcal{B}$ satisfy $E_\alpha B = B = BE_\alpha$. Due to the topological fullness of $_{\boldsymbol{\mathcal{B}}}\langle\cdot,\cdot\rangle$ we find $\boldsymbol{x}_i, \boldsymbol{y}_i \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ such that $E_\alpha = \sum_i {}_{\boldsymbol{\mathcal{B}}}\langle \boldsymbol{x}_i, \boldsymbol{y}_i\rangle + \lambda C$ with some element $C \in \boldsymbol{\mathcal{B}}$. Then $B = \sum_i {}_{\boldsymbol{\mathcal{B}}}\langle \mathsf{L}_{\boldsymbol{\mathcal{B}}}(B_0)\boldsymbol{x}_i, \boldsymbol{y}_i\rangle + \lambda BC' = \lambda C''$ shows that B cannot have a zeroth order and hence $B = 0$. Now let $\boldsymbol{x} \in \boldsymbol{\mathfrak{X}}_L$ and $\boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{\mathfrak{X}}$ then $\mathsf{L}_{\boldsymbol{\mathcal{B}}}({}_{\boldsymbol{\mathcal{B}}}\langle \boldsymbol{y}, \boldsymbol{x}\rangle)\boldsymbol{z} = \boldsymbol{y}\mathsf{R}_{\boldsymbol{\mathcal{A}}}(\langle \boldsymbol{x}, \boldsymbol{z}\rangle_{\boldsymbol{\mathcal{A}}}) \in \lambda_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ implies that $B := \mathfrak{C}_{\boldsymbol{\mathcal{B}}}\langle \boldsymbol{y}, \boldsymbol{x}\rangle$ satisfies $\mathsf{L}_{\boldsymbol{\mathcal{B}}}(B)\,{}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} \subseteq \lambda\,{}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ whence $B = 0$. Thus $\boldsymbol{x} \in {}_L\boldsymbol{\mathfrak{X}}$ follows. Reverting the argument finishes the proof. □

Thus the classical limit $_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = \mathfrak{C}\,{}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} = {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}/\boldsymbol{\mathfrak{X}}_L$ is a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule and inherits a $\mathcal{B}$-valued and $\mathcal{A}$-valued inner product. In order to guarantee that $_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is indeed an equivalence bimodule we have to guarantee the positivity requirements (X4) and (Y4) as well as (P) and (Q). For the first two, it is sufficient to consider positive deformations $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively. For the second two, we can either impose the stronger conditions (P1)–(P3) and (Q1)–(Q3) which behave well under the classical limit or we have to know more on the deformability of *-representations. For the next theorem we shall assume that we are able to guarantee (P) and (Q) directly.

Theorem 9.7. *Let* $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}$ *be positive deformations of* **-algebras* $\mathcal{A}, \mathcal{B}$ *over* C *with approximate identities and let* $_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *be a* $(\boldsymbol{\mathcal{B}}\text{-}\boldsymbol{\mathcal{A}})$*-equivalence bimodule* (*where we actually only need* (tX6) *and* (tY6)). *If the classical limit* $_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} = {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}/\boldsymbol{\mathfrak{X}}_L$ *satisfies* (P) *and* (Q) *then* $_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *is a* $(\mathcal{B}\text{-}\mathcal{A})$*-equivalence bimodule. If* $_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ *satisfies in addition* (P1)–(P3) *and* (Q1)–(Q3) *then the classical limit is automatically an equivalence bimodule also satisfying* (P1)–(P3) *and* (Q1)–(Q3).

Proof. It remains to show (E3) for the classical limit which is a simple computation. □

We shall now discuss some more particular cases. First we can consider a bimodule for the deformed algebras of the more particular form $_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} = \mathfrak{X}[[\lambda]]$, where $\mathfrak{X}$ is a C-module. From the deformation point of view this is a natural restriction. In this case we can use the λ-adic topology of $\mathfrak{X}[[\lambda]]$ to define also topological versions of the conditions (P1)–(P3), which are slightly weaker:

(tP1) There exist $\mathsf{C}[[\lambda]]$-submodules $\boldsymbol{\mathfrak{X}}^{(i)} \subseteq {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$, $i \in I$, such that $\boldsymbol{\mathfrak{X}}^{(i)} \perp \boldsymbol{\mathfrak{X}}^{(j)}$ for all $i \neq j \in I$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ and $\oplus_{i\in I}\boldsymbol{\mathfrak{X}}^{(i)}$ is λ-adically dense in $_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} = \mathfrak{X}[[\lambda]]$.

(tP2) The $\boldsymbol{\mathcal{A}}$-right action $\boldsymbol{\mathfrak{R}}_{\mathcal{A}}$ preserves this direct sum.

(tP3) Each $\boldsymbol{\mathfrak{X}}^{(i)}$ is topologically pseudo-cyclic for $\boldsymbol{\mathfrak{R}}_{\mathcal{A}}$, i.e. there exist directed submodules $\{\boldsymbol{\mathfrak{X}}^{(i)}_\alpha\}_{\alpha\in I^{(i)}}$ with pseudo-cyclic vectors $\boldsymbol{\Omega}^{(i)}_\alpha$ such that $\bigcup_{\alpha\in I^{(i)}}\boldsymbol{\mathfrak{X}}^{(i)}_\alpha$ is λ-adically dense in $\boldsymbol{\mathfrak{X}}^{(i)}$.

An easy check similar to the proof of Lemma 3.1 ensures that (tP1)–(tP3) still imply (P). Then the next lemma is shown straightforwardly using analogous arguments as in the proof of Lemma 8.1.

Lemma 9.8. *If in addition, the bimodule is of the form* ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} = \mathfrak{X}[[\lambda]]$ *and satisfies* (tP1)–(tP3) *then the classical limit* ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *satisfies* (P1)–(P3).

The other important case is when the *-algebras $\mathcal{A}$ and $\mathcal{B}$ have sufficiently many positive linear functionals (and approximate identities) and when we consider positive deformations $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ which is the case in deformation quantization. Then one can characterize the space $\boldsymbol{\mathfrak{X}}_L$ as in the case of pre-Hilbert spaces by the following lemma.

Lemma 9.9. *The space* $\boldsymbol{\mathfrak{X}}_L$ *coincides with* $\{\boldsymbol{x} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} | \mathfrak{C}\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{A}}} = 0\}$.

Proof. One inclusion is trivial. For the other we consider $\boldsymbol{x}, \boldsymbol{y} \in {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$, then we have for all positive $\mathsf{C}[[\lambda]]$-linear functionals $\omega : \boldsymbol{\mathcal{A}} \to \mathsf{C}[[\lambda]]$ the inequality $\omega(\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}})\overline{\omega(\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}})} \leq \omega(\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{A}}})\omega(\langle \boldsymbol{y}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}})$. Hence we obtain in the classical limit

$$\omega_0(\mathfrak{C}\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}})\overline{\omega_0(\mathfrak{C}\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}})} \leq \omega_0(\mathfrak{C}\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{A}}})\omega_0(\mathfrak{C}\langle \boldsymbol{y}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}}),$$

where $\omega_0 = \mathfrak{C}\omega$ is the classical limit of ω. If $\mathfrak{C}\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{A}}} = 0$ then $\omega_0(\mathfrak{C}\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{A}}}) = 0$ follows. Since $\boldsymbol{\mathcal{A}}$ is a positive deformation any positive linear functional of $\mathcal{A}$ occurs as classical limit of some ω and since $\mathcal{A}$ has sufficiently many positive linear functionals and an approximate identity it follows from Proposition 2.8 that $\mathfrak{C}\langle x, y\rangle_{\mathcal{A}} = 0$. □

Thus in this case we automatically end up with a classical limit ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ of ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ which satisfies (X4′). Hence, in the case of an equivalence bimodule ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ we obtain a *non-degenerate* equivalence bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ in the classical limit (whenever we can guarantee (P) and (Q) for ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$).

As an application to deformation quantization we observe that $C_0^\infty(M)$ as well as $C^\infty(M)$ have sufficiently many positive linear functionals (use the δ-functionals) as well as approximate identities, and that star products on symplectic manifolds are positive deformations, see [23, Proposition 5.1] and Corollary B.5. Thus we are in the 'optimal' situation in this case. Nevertheless, to show that formally Morita equivalent star products imply diffeomorphic underlying manifolds, we essentially do not need any positivity requirements. In fact, we only need a bimodule satisfying (E1)–(E3) without (X4) and (Y4) for the quantized algebras in order to obtain a bimodule satisfying (E1)–(E3) (without (X4) and (Y4)) in the classical limit. This is already sufficient to guarantee that the underlying manifolds are diffeomorphic according to the results on commutative *-algebras in Section 7 whence we can state this result for arbitrary Poisson manifolds.

Corollary 9.10. *Let* $(M, *)$ *and* $(\tilde{M}, \tilde{*})$ *be Poisson manifolds with Hermitian star products such that for* $(C^\infty(M)[[\lambda]], *)$ *and* $(C^\infty(\tilde{M})[[\lambda]], \tilde{*})$ *there exists a bimodule satisfying* (E1)–(E3) (*not necessarily* (X4) *and* (Y4)). *Then* M *and* $\tilde{M}$ *are diffeomorphic*.

In particular the above corollary gives an 'asymptotic' explanation why Morita equivalent (in the C^*-algebraic sense) quantum tori have to have at least the same classical dimension, see also [69] for a more sophisticated discussion on the Morita equivalence of quantum tori.

Let us now conclude with a few remarks on the 'reverse' question, namely of deformation of bimodules. Assume that two *-algebra deformations $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ of two *-algebras $\mathcal{A}$, $\mathcal{B}$ over C are given and let furthermore a ($\mathcal{B}$-$\mathcal{A}$)-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ with $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ with some properties like, e.g. (X1)–(X5), (P), or (P1)–(P3) be given. Then a ($\boldsymbol{\mathcal{B}}$-$\boldsymbol{\mathcal{A}}$)-*bimodule deformation* of ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is a ($\boldsymbol{\mathcal{B}}$-$\boldsymbol{\mathcal{A}}$)-bimodule ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ with $\boldsymbol{\mathcal{A}}$-valued inner product, having the same properties, such that the classical limit of ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}}$ is ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. More restrictively, one can demand that ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathfrak{X}}_{\boldsymbol{\mathcal{A}}} = {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}[[\lambda]]$ as $\mathsf{C}[[\lambda]]$-module.

In general the question of existence of such a deformation is very hard to attack: for the deformation of the bimodule structure alone one can apply the usual cohomological techniques which are already rather complicated as we have to deal with a bimodule instead of a module. Thus the Hochschild cohomology of $\mathcal{B}\otimes\mathcal{A}^{\mathrm{op}}$ with values in ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ viewed as $\mathcal{B}\otimes\mathcal{A}^{\mathrm{op}}$-module becomes relevant. But since we also want an $\boldsymbol{\mathcal{A}}$-valued inner product one has even more obstructions as one wants *positivity* of this inner product. Thus the *inequalities* occurring in the positivity requirements do not seem to permit a cohomological approach and thus one has to develop further techniques in order to deal with this question.

Another question concerning the deformations of such bimodules is the uniqueness of the deformations: here one has to develop a reasonable notion of 'equivalence of deformations'. One possibility is that one calls two deformations of ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ *functorially equivalent* if the corresponding functors of algebraic Rieffel induction are naturally isomorphic. We shall leave these questions to future work and discuss only one example based on Proposition 4.8.

Let $\Phi:\mathcal{B}\to\mathcal{A}$ be a *-homomorphism of *-algebras over C and let *-algebra deformations $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively, be given. Then we consider the ($\mathcal{B}$-$\mathcal{A}$)-bimodule ${}_{\Phi(\mathcal{B})}\mathcal{A}_{\mathcal{A}}$ with the $\mathcal{A}$-valued inner product as in (4.7). If we are able to find a deformation $\boldsymbol{\Phi}=\sum_{r=0}^{\infty}\lambda^r\boldsymbol{\Phi}_r$ of $\Phi=\boldsymbol{\Phi}_0$ into a *-homomorphism $\boldsymbol{\Phi}:\boldsymbol{\mathcal{B}}\to\boldsymbol{\mathcal{A}}$ of the deformed algebras then it is an easy check that the corresponding bimodule ${}_{\boldsymbol{\Phi}(\boldsymbol{\mathcal{B}})}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{A}}}$ is a deformation of ${}_{\Phi(\mathcal{B})}\mathcal{A}_{\mathcal{A}}$: in this case the complicated question of the positivity properties (X4) and (P) is trivially answered by Proposition 4.8 and we are 'only' faced with the cohomological problem of finding a deformation of a *-homomorphism, which is of course still complicated enough.

Proposition 9.11. *Let $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ be *-algebra deformations of *-algebras $\mathcal{A}$ and $\mathcal{B}$ over* C *and let $\boldsymbol{\Phi}:\boldsymbol{\mathcal{B}}\to\boldsymbol{\mathcal{A}}$ be a *-homomorphism. Then ${}_{\boldsymbol{\Phi}(\boldsymbol{\mathcal{B}})}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{A}}}$ is a deformation of ${}_{\Phi(\mathcal{B})}\mathcal{A}_{\mathcal{A}}$, where $\Phi:\mathcal{B}\to\mathcal{A}$ is the classical limit of $\boldsymbol{\Phi}$.*

10. Conclusion and further questions

We shall conclude this work with some final remarks and additional questions arising from our approach to Rieffel induction and Morita equivalence. We hope a foundation has been laid for further investigations and applications of these ideas, which we plan to study in the future.

First of all, the relation of the original notion of Rieffel induction and Morita equivalence for C^*-algebras to our more algebraic point of view needs further study. Many of our algebraic results, including the proofs, are motivated by the C^*-algebraic case so it would be interesting to see to what extent further results can be carried to the purely algebraic framework. First steps into this direction are done in [23]. On one hand, this can help understanding what is particular to C^*-algebras and, on the other hand, one could make many of the C^*-algebra results available also for other *-algebras, which is interesting from the mathematical and physical points of view. In particular, studying *-algebras over $\mathsf{C} = \mathbb{C}[[\lambda]]$ is of special interest, as this ring governs various asymptotic situations in physics: the formal parameter λ could correspond to Planck's constant $\hbar$ as in deformation quantization but also to a coupling constant α as in various versions of perturbation theory (see, e.g., [34,35] for recent usage of the order structure of $\mathbb{R}[[\lambda]]$ in the context of quantum field theory). A better understanding of concrete connections between formal and C^*-algebraic Morita equivalence would be of special interest for the case of quantum tori, since they have gained increasing attention due to their relation to string and M theories, see, e.g., [30,69,71]. It seems reasonable to apply the asymptotic approach using $\mathbb{C}[[\lambda]]$ to this example, since the quantum tori are entirely determined by their classical, flat Poisson structure on T^n and the corresponding Weyl–Moyal star product.

Second, again motivated by deformation quantization, one can try to develop topological versions of the constructions in this paper 'in between' the purely algebraic context and the C^*-algebraic case. In deformation quantization, it seems that the locally convex topologies of smooth functions are 'closer' to the formal approach than the C^*-norm based topologies, examples can be found in [15–17,62,63]. Thus it seems reasonable to use these intermediate topologies to handle the convergence problems of formal deformations. One can also use the canonical order topology of the underlying ordered ring to develop a 'non-Archimedian functional analysis', a point of view taken, e.g. in [21] and references therein.

Third, from a more geometrical point of view, one should compare Xu's notion of Morita equivalence for Poisson manifolds with our notion for star products. At first glance, one is tempted to view Xu's notion as the 'first non-trivial order' of deformation quantization. However, Corollary 9.10 and [80] (see also, e.g., [28, Proposition 8.6]) show that, at least in this naive way, this is not the case. So the possible relation between these ideas need further study. More generally, one could try to use the algebraic framework, especially for $\mathsf{C} = \mathbb{C}[[\lambda]]$, to establish asymptotic analogues of quantum geometry in the spirit of Connes' non-commutative geometry [29] and study (semi-)classical limits. Physically, λ could play here the role of a parameter associated to the Planck scale.

Fourth, there arise several natural questions within the framework of deformation quantization. Most important is the task to determine the equivalence classes of Morita equivalent star products (note that Theorem 9.7 suggests that the underlying manifold has to be the same). We observe that Rieffel induction alone is of great interest as it may provide a way for quantizing phase space reduction from the viewpoint of states and representations. While the reduction of the related observable algebras is quite well-understood in the most important cases [14,40], a formulation for the states is still missing. We also remark that Landsman uses Rieffel induction within the C^*-algebraic framework to formulate analogues of phase space reduction, see [51] and references therein. Again our approach seems to be most suited to formulate an asymptotic analogue filling the gaps between [14] and [51]. Finally, the relation between formal Morita equivalence and the locality structures as discussed in [75] should be investigated and results like Proposition 4.2 should be further explored in this context.

Fifth, there are further physical applications where the asymptotic point of view can be used. We can mention here the WKB approximation scheme (as well as the closely related short wave approximation in theoretical optics), see, e.g., [8]. It is not surprising, due to the asymptotic character of this method, that it admits a formulation within the framework of formal deformation quantization, see [17,20]. In particular, it seems possible to use our results of Section 6 to find a transition from [17,20] to endomorphism-valued Hamiltonians as discussed, e.g. in [36,37].

Finally, we mention some purely algebraic open questions. It would be interesting to find more examples or counter-examples which illustrate how strong the notion of formal Morita equivalence is. First, one could try to find an example of two *-algebras with sufficiently many positive linear functionals and approximate identities which have equivalent categories of strongly non-degenerate *-representations but no equivalence bimodule. Recall that our example in Corollary 5.20 uses the Grassmann algebra which has ‘few’ positive linear functionals. A more general class of examples based on this idea is discussed in [44]. Second, we have not addressed the question of how the lattices of ideals (or *-ideals) are related for formally Morita equivalent *-algebras. As in the case of C^*-algebras, one can prove that for formally Morita equivalent *-algebras, the lattices of closed *-ideals are isomorphic. Here a *-ideal is called closed if it occurs as the kernel of a *-representation, see [44]. In [44] we compute further ‘invariants’ of formal Morita equivalence. In the case of deformation quantization, one can even imagine obtaining finer results by considering the locality structure as in [75]. It appears that for the question of formal Morita-invariants, the positivity requirements ((E4), (X4), (Y4)) might play only a minor role and thus one should consider bimodules not necessarily fulfilling them (see note below). Perhaps one is able to show the positivity requirements directly for some cases (at least for strongly non-degenerate *-representations), as this is possible for C^*-algebras.

Note. After the completion of this article, Prof. Ara brought his work [4,5] to our attention. In [5], Ara develops the notion of Morita equivalence for (non-degenerate and idempotent) rings with involution (called Morita *-equivalence), which encompasses the notion of formal Morita equivalence as defined here. For these rings, Ara considers suitable categories of modules, studies certain types of (pairs of) functors defining equivalence of these categories and succeeds in proving a Morita-like theorem that characterizes these functors in terms of the existence of ’inner product bimodules’ (which are essentially our equivalence bimodules without the positivity requirements (X4), (Y4), (E4)). Ara also shows that

Morita *-equivalent rings have *-isomorphic centroids and, as a consequence, that Morita *-equivalence implies *-isomorphism for commutative rings. His results hold in our setting, that is, for *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, provided one assumes the existence of approximate identities (to make the *-algebras non-degenerate and idempotent), and in particular can be used to extend Proposition 7.6 and Corollary 7.7 to non-unital situations. However, the notion of positivity, which is crucial throughout the present paper, is absent in Ara's approach. Moreover, several constructions and results presented here do not assume the *-algebras to be non-degenerate or idempotent. In [44] we investigate the relations between these two approaches more closely.

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Appendix A. Positive matrices over ordered rings

In this appendix we collect some results on positive matrices in $M_n(\mathsf{C})$, where $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with an ordered ring R. The main point we want to emphasize is that almost all results on positive matrices known from $M_n(\mathbb{C})$ can be carried over to this more general situation if one avoids the notion of square roots in the proofs.

Consider the free C-module C^n with canonical basis $e_1, \ldots, e_n$ and define the usual Hermitian product as in Section 6, where we have seen that $M_n(\mathsf{C})$ coincides with $\mathfrak{B}(\mathsf{C}^n)$ after the usual identification with $\mathrm{End}_{\mathsf{C}}(\mathsf{C}^n)$. Thus $M_n(\mathsf{C})$ becomes a *-algebra in the usual way and we want to study the positive linear functionals and the positive elements of $M_n(\mathsf{C})$. Since $M_n(\mathsf{C})$ is a free module any linear functional $\omega : M_n(\mathsf{C}) \to \mathsf{C}$ can be written in the form

$$\omega : A \mapsto \omega(A) = \mathrm{tr}(\varrho A), \quad \text{with} \quad \varrho \in M_n(\mathsf{C}), \tag{A.1}$$

using the trace functional tr. Clearly ω is a real functional if and only if $\varrho = \varrho^*$. As a positive functional is necessarily real (since $M_n(\mathsf{C})$ has a unit element) we restrict ourselves to Hermitian matrices ϱ from now on.

Lemma A.1. *Let* $\varrho = \varrho^* \in M_n(\mathsf{C})$ *then* $\mathrm{tr}(\varrho A^* A) \geq 0$ *for all* $A \in M_n(\mathsf{C})$ *if and only if* $\langle v, \varrho v \rangle \geq 0$ *for all* $v \in \mathsf{C}^n$.

Proof. If $\langle v, \varrho v \rangle \geq 0$ for all $v \in \mathsf{C}^n$ then consider $v_i^{(k)} := \overline{A_{ki}}$ whence $\mathrm{tr}(\varrho A^* A) = \sum_k \langle v^{(k)}, \varrho v^{(k)} \rangle \geq 0$ for all $A \in M_n(\mathsf{C})$ follows. If on the other hand $\mathrm{tr}(\varrho A^* A) \geq 0$ for all A then choose A with $A_{ki} = \overline{v}_i$ for $k = 1, \ldots, n$. Then $\mathrm{tr}(\varrho A^* A) = \sum_k \langle v, \varrho v \rangle =$

$n\langle v, \varrho v\rangle \geq 0$. But since R has characteristic zero and $n > 0$ we conclude $\langle v, \varrho v\rangle \geq 0$. $\square$

We call a Hermitian matrix ϱ satisfying $\langle v, \varrho v\rangle \geq 0$ for all $v \in \mathsf{C}^n$ a *density matrix*, and hence we have established a one-to-one correspondence between density matrices and positive linear functionals of $M_n(\mathsf{C})$.

In order to characterize the positive elements in $M_n(\mathsf{C})$ we have first to pass to the quotient fields $\hat{\mathsf{R}}$ and $\hat{\mathsf{C}}$ of R and C. Remember that $\hat{\mathsf{R}}$ is an ordered field such that $\mathsf{R} \hookrightarrow \hat{\mathsf{R}}$ is order preserving, and canonically one has $\hat{\mathsf{C}} \cong \hat{\mathsf{R}}(\mathrm{i})$. Then the canonical inclusion $M_n(\mathsf{C}) \hookrightarrow M_n(\hat{\mathsf{C}})$ is an injective *-homomorphism of *-algebras over C. The following lemma shows that a density matrix $\varrho \in M_n(\mathsf{C})$ is still a density matrix in $M_n(\hat{\mathsf{C}})$.

Lemma A.2. *Let $\varrho \in M_n(\mathsf{C})$ be a Hermitian matrix. Then $\langle v, \varrho v\rangle \geq 0$ for all $v \in \mathsf{C}^n$ if and only if $\langle \hat{v}, \varrho \hat{v}\rangle \geq 0$ for all $\hat{v} \in \hat{\mathsf{C}}^n$.*

Proof. The proof is obtained by observing that for finitely many elements $\hat{v}_i \in \hat{\mathsf{C}}$, written as fractions, we can find a common denominator which we can choose real and positive. $\square$

Lemma A.3. *Let $\varrho \in M_n(\hat{\mathsf{C}})$ be a density matrix. Then there exists a basis $v_1, \ldots, v_n$ of $\hat{\mathsf{C}}^n$ and non-negative numbers $p_1, \ldots, p_n \in \hat{\mathsf{R}}$ such that $\langle v_i, \varrho v_j\rangle = \delta_{ij} p_i$ for all i, j.*

Proof. This is standard, see, e.g., [45, Theorem 6.19], where $p_i \geq 0$ follows from $p_i = \langle v_i, \varrho v_i\rangle \geq 0$. $\square$

Note that for the above lemma we have to use the quotient fields $\hat{\mathsf{R}}$ and $\hat{\mathsf{C}}$ instead of R and C. Denoting by $U \in M_n(\hat{\mathsf{C}})$ the invertible matrix of the basis transformation, i.e. $e_i = U v_i$ for $i = 1, \ldots, n$, we obtain the following form of ϱ

$$\varrho = \sum_i p_i U^* P_i U = \sum_i p_i U^* P_i^* P_i U \in M_n(\hat{\mathsf{C}})^{++}, \tag{A.2}$$

where $P_i = P_i^* = P_i^2 \in M_n(\hat{\mathsf{C}})$ is the matrix such that $P_i v = \langle e_i, v\rangle e_i$. Note that U is *not* unitary in general. Nevertheless we can use (A.2) to prove the following proposition.

Proposition A.4. *Let $A \in M_n(\mathsf{C})$ be Hermitian. Then A is positive if and only if A is a density matrix, i.e. $\langle v, Av\rangle \geq 0$ for all $v \in \mathsf{C}^n$.*

Proof. If A is positive then clearly $\langle v, Av\rangle \geq 0$ for all $v \in \mathsf{C}^n$ since the functional $A \mapsto \langle v, Av\rangle$ is a positive linear functional. For the other direction we have to show $\omega(A) \geq 0$ for all positive linear functionals $\omega : M_n(\mathsf{C}) \to \mathsf{C}$. Due to Lemma A.1 we have to show $\mathrm{tr}(\varrho A) \geq 0$ for all density matrices $\varrho \in M_n(\mathsf{C})$, and Lemma A.2 allows us to consider $\hat{\mathsf{C}}$ instead of C. Then $\mathrm{tr}(\varrho A) = \sum_i p_i \mathrm{tr}(U^* P_i U A) = \sum_i p_i \langle U^* e_i, A U^* e_i\rangle \geq 0$ proves the proposition. $\square$

As a remark we would like to mention that if $\hat{\mathsf{R}}$ is a real closed field then $\hat{\mathsf{C}}$ is algebraically closed and thus any density matrix $\varrho \in M_n(\hat{\mathsf{C}})$ can be diagonalized by a unitary matrix with positive eigenvalues $\lambda_i \geq 0$. Thus $\mathrm{tr}(\varrho A)$ can be computed in the eigenbasis of ϱ, simplifying the proof. On the other hand, an analogue of Lemma A.2 is not necessarily true if the quotient fields are replace by the real and algebraic closures, respectively. In case $\mathsf{R} = \mathbb{Z}$, $\hat{\mathsf{R}} = \mathbb{Q}$ and the real and algebraic closures of $\mathbb{Q}$, a simple continuity argument proves an analogue of Lemma A.2 since $\mathbb{Q}$ is dense in its real closure with respect to the order topology. But in general this is no longer true, e.g. the field of formal Laurent series $\mathbb{R}((\lambda))$ is not dense with respect to the order topology in its real closure $\mathbb{R}\langle\langle\lambda^*\rangle\rangle$, the field of formal Newton–Puiseux series, see, e.g., [17,21]. Let us finally mention the following corollaries.

Corollary A.5. *Let $A, B \in M_n(\mathsf{C})$ be positive matrices then* $\mathrm{tr}(AB) \geq 0$.

Corollary A.6. *Let $A \in M_n(\mathsf{C})^+$ then $A \in M_n(\hat{\mathsf{C}})^{++}$.*

Corollary A.7. *Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be two C-modules with positive semi-definite Hermitian products. Then $\langle \phi \otimes \psi, \phi' \otimes \psi' \rangle_{\mathfrak{H}} := \langle \phi, \phi' \rangle_1 \langle \psi, \psi' \rangle_2$ extends to a positive semi-definite Hermitian product on $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$.*

Proof. Let $\chi = \phi_1 \otimes \psi_1 + \cdots + \phi_n \otimes \psi_n \in \mathfrak{H}$ then $\langle \chi, \chi \rangle = \mathrm{tr}(MN)$, where $M, N \in M_n(\mathsf{C})$ are given by their matrix elements $M_{ij} = \langle \phi_i, \phi_j \rangle_1$ and $N = \langle \psi_j, \psi_i \rangle_2$. Clearly M, N are positive matrices since $\langle v, Mv \rangle = \langle \phi, \phi \rangle \geq 0$ where $\phi = v_1\phi_1 + \cdots + v_n\phi_n$ and $v \in \mathsf{C}^n$, and similar for N. Then $\langle \chi, \chi \rangle_{\mathfrak{H}} \geq 0$ by Corollary A.5. $\square$

Appendix B. Positive linear functionals for $C_0^\infty(M)$ and $C^\infty(M)$

As deformation quantization is our main motivation, we shall use this appendix to describe its classical limit and show that for $C_0^\infty(M)$ and $C^\infty(M)$, our characterization of positive linear functionals and algebra elements yields the expected results. Some subtleties arise as the Riesz' representation theorem, which essentially governs the situation, is usually only considered in the continuous category while we have to work in the smooth category using also functions with non-compact support. Thus the following lemmas, which should be well-known, can be viewed as 'positivity implies continuity' statements in the smooth category.

Lemma B.1. *Let $\omega : C_0^\infty(M) \oplus \mathbb{C}1 \to \mathbb{C}$ be a positive linear functional. Then ω is continuous with respect to the sup-norm, i.e. $|\omega(f)| \leq \omega(1)\|f\|_\infty$ for all $f \in C_0^\infty(M) \oplus \mathbb{C}1$.*

Proof. Let $f \in C_0^\infty(M) \oplus \mathbb{C}1$ then $\|f\|_\infty^2 1 - \overline{f}f \in C_0^\infty(M) \oplus \mathbb{C}1$ is non-negative whence for all $\varepsilon > 0$ the function $(\|f\|_\infty^2 + \epsilon)1 - \overline{f}f$ is strictly positive. Thus the square root is still smooth and contained in $C_0^\infty(M) \oplus \mathbb{C}1$ whence $\omega((\|f\|_\infty^2 + \epsilon)1 - \overline{f}f) \geq 0$.

Thus $\omega(\overline{f}f) \leq \|f\|_\infty^2 \omega(1)$ follows and with the Cauchy–Schwarz inequality $|\omega(f)|^2 \leq \omega(\overline{f}f)\omega(1) \leq \|f\|_\infty^2 \omega(1)^2$, the proof is finished. □

Thus ω extends uniquely to the C^*-algebra completion of $C_0^\infty(M) \oplus \mathbb{C}1$ and by Riesz' representation theorem, see, e.g., [70, p. 40], we conclude that ω is given by a positive measure of finite volume given by $\omega(1)$.

If we now consider $C_0^\infty(M)$ instead, then a positive linear functional needs no longer to be continuous in the sup-norm, take, e.g. $M = \mathbb{R}$ and $f \mapsto \int_\mathbb{R} f(x)x^2\,\mathrm{d}x$, but 'locally' this is still true: choose an approximate identity $\{O_n, \chi_n\}_{n\in\mathbb{N}}$ and let $\omega : C_0^\infty(M) \to \mathbb{C}$ be a positive linear functional, then $\omega_n(f) := \omega(\chi_n f \chi_n)$ is still positive and has compact support in O_{n+1} such that the restrictions of ω and ω_n on $C_0^\infty(O_n)$ coincide. A simple computation shows that ω_n can now be extended in a unique way to a well-defined positive linear functional of $C_0^\infty(M) \oplus \mathbb{C}1$ by setting $\omega_n(1) = \omega(\chi_n\chi_n)$ whence we can apply the last lemma. Thus ω_n is given by a positive measure having compact support in O_{n+1} and we thus conclude the following lemma.

Lemma B.2. *Let $\omega : C_0^\infty(M) \to \mathbb{C}$ be a positive linear functional. Then ω is given by a positive measure with finite volume for all compact subsets of M.*

Finally, consider a positive linear functional $\omega : C^\infty(M) \to \mathbb{C}$ and let $f \in C^\infty(M)$. By the Cauchy–Schwarz inequality we find $|\omega((1-\chi_n)f)|^2 \leq \omega((1-\chi_n)(1-\chi_n))\omega(\overline{f}f)$, where we used again an approximate identity. But since $1-\chi_n \in C_0^\infty(M) \oplus \mathbb{C}1$ we can apply Lemma B.1 whence in particular $\omega((1-\chi_n)(1-\chi_n)) \to 0$ as $n \to \infty$. Thus $\omega((1-\chi_n)f) \to 0$, too, and hence $\omega(\chi_n f) \to \omega(f)$. Thus ω is completely determined by its restriction to $C_0^\infty(M) \oplus \mathbb{C}1$. In the case where M is non-compact we find 'sufficiently unbounded' functions $f \in C^\infty(M)$ to conclude that the measure actually has not only finite volume but even compact support.

Lemma B.3. *Let $\omega : C^\infty(M) \to \mathbb{C}$ be a positive linear functional. Then ω is given by a positive measure with compact support.*

Since the δ-functionals are clearly positive linear functionals it follows that $f(x) \geq 0$ for all $x \in M$ is a necessary condition for a function to be a positive algebra element in sense of Definition 2.3. The above form of the positive linear functionals of $C^\infty(M)$ or $C_0^\infty(M)$ shows that this is also sufficient, as one would expect the corollary.

Corollary B.4. *$f \in C^\infty(M)^+$ (or $C_0^\infty(M)^+$) if and only if $f(x) \geq 0$ for all $x \in M$.*

Corollary B.5. *Let $(M, *)$ be a symplectic manifold with Hermitian star product. Then the algebra $(C^\infty(M)[[\lambda]], *)$ is a positive deformation.*

Proof. The case $(C_0^\infty(M)[[\lambda]], *)$ was shown in [23, Proposition 5.1]. Since any positive linear functional of $C^\infty(M)$ is given by a positive linear functional of $C_0^\infty(M)$ having compact support and since the construction in [23, Proposition 5.1] does not increase the support, the corollary follows. □

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